Teacher Notes

T 1
Types of Triangles
Teacher Notes

T 1
Types of Triangles

Task: To develop definitions for right, acute, obtuse, isosceles, and equilateral triangles.

Procedure:
- Construct a random right triangle.
- Make measurements on the triangle.
- Record your drawing of the triangle and its measurements in the space provided.
- Repeat this process with other right triangles.
- Based on your findings, develop a conjecture about right triangles and write it below.
- Repeat this procedure for acute, obtuse, isosceles, and equilateral triangles.

Drawings & Data

Conjectures

A right triangle is a triangle which... ______

An acute triangle is a triangle which... ______

An obtuse triangle is a triangle which... ______

An isosceles triangle is a triangle which... ______

An equilateral triangle is a triangle which... ______

Notes

This exercise is designed to give students an opportunity to use their own ideas and words in defining the five different types of triangles. Don't expect students to develop precise definitions. They should start to see patterns emerge as they measure the sides and the angles of many different examples of the same type of triangle. After students have worked in pairs, the findings should be presented and reviewed by the class as a whole. This review provides an opportunity to talk about what it means to write a "good" definition, and to arrive at some consensus on definitions for the different types of triangles.

This process of arriving at consensus can be an interesting one. Take, for example, the definition of isosceles triangle. Some students will claim that an isosceles triangle has two sides of equal length; some will claim two angles of equal measure; and still others will claim two sides of equal length <u>and</u> two sides of equal measure. Defending these positions and unravelling and coalescing various points of view with a class produces a lively and instructive discussion.

Comments

T 2
Angle Measurements
Teacher Notes

T 2
Angle Measurements

Task: To explore the relationship among the interior angles in different types of triangles.

Procedure:
- Construct a triangle.
- Measure each angle.
- Draw the triangles and record the angle measurements on the chart below.
- Repeat this procedure on five other triangles.
- On the following page, state conjectures about your findings.

Triangle Drawings	∠ABC	∠BCA	∠CAB
1.			
2.			
3.			
4.			
5.			
6.			

T 2 (page 2)
Angle Measurements

Conjectures

State a conjecture about the sum of the three angles.

State other conjectures about the three angles or the relationship between the sum of two angles and the third angle.

Conjectures

- The sum of the measures of the angles in any triangle is 180°.
- In a right triangle, the sum of the measures of the two non-right angles is 90°.
- In an obtuse isosceles triangle, the sum of the measures of the two acute angles is less than 90°.
- In a triangle, the measures of the three angles may be equal (60°), different (40°, 60°, 80°), or two angles may be equal and the third different (40°, 40°, 100°).

Notes

Data are the "stuff" from which conjectures are made and in the case of the SUPPOSER that means drawings and measurements. Early on in working with the SUPPOSER, emphasize to students that measurements and fairly accurate freehand drawings are essential tools for supporting and defending their geometric ideas. Also, it is a good idea in the early going to provide students with a structure for collecting their data, *i.e.,* a table with labeled columns, and perhaps to consider filling in a table partially when you think that the problem is a difficult one.

Comments

T 3 and T 4
Interior and Exterior Angles
Exterior Angles
Teacher Notes

T 3
Interior and Exterior Angles

Task: To explore the relationship among the three interior angles and one exterior angle in different types of triangles.

Procedure:
- Construct an acute $\triangle ABC$.
- Draw an extension of side $\overline{BA}$ such that BA = AD.
- Measure the angles and record the measurements in the chart below.
- Repeat the steps for other types of triangles. Use the Repeat option.
- For each triangle, measure the interior angles of $\triangle ABC$ and the exterior $\angle CAD$.
- State your conjectures about the relationship between the exterior and interior angles.

Diagram:
On an acute triangle, $\overline{BA}$ is extended from A such that BA = AD.
$\angle CAD$ created by the extension is called an exterior angle.

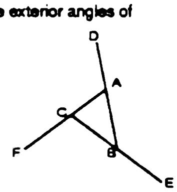
Triangles	exterior ∠CAD	∠ABC	∠BCA	∠CAB
Right				
Acute				
Obtuse				
Isosceles				
Equilateral				

Conjectures

T 4
Exterior Angles

Task: To investigate the relationship among the three exterior angles of triangles.

Procedure:
- Construct an acute triangle.
- Draw the three exterior angles.
- Measure the exterior angles.
- Record drawings and measurements.
- Repeat the steps for other types of triangles.
- State your conjectures.



Drawings & Data

Conjectures

What is the sum of the measures of the three exterior angles for an acute triangle? __________

Is this sum the same for all types of triangles? __________

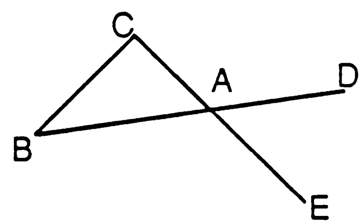
Conjectures

T 3

- $\angle CAD = \angle ACB + \angle CBA$
- For acute triangles only, the measure of each exterior angle is greater than any interior angle.

Note: Actually, for any $\triangle ABC$, there are two exterior angles for each angle in the triangle. For example, $\angle CAD$ and $\angle BAE$ are both exterior angles for $\angle A$ in $\triangle ABC$. $\overline{BA}$ is extended to create $\angle CAD$; $\overline{CA}$ is extended to create $\angle BAE$.

However, since $\angle CAD$ and $\angle BAE$ are vertical angles, $\angle CAD = \angle BAE$. In this problem, and most others in geometry, we are interested only in the relationships involving one exterior angle at each vertex.



T 4

- The sum of the measures of the three exterior angles is 360°.
- Yes, it is the same for all types of triangles.

Comments

T 5
Triangles Whose Sides are Three Consecutive Integers
Teacher Notes

T 5

Triangles Whose Sides Are Three Consecutive Integers

Task: To discover whether sides whose lengths are three consecutive integers form triangles. When they do, to investigate the types of triangles they form.

Procedure:
- Construct a triangle using the side-side-side option.
- Use side lengths that are consecutive integers (*e.g.*, 4-5-6).
- Record your drawing of the triangle and the lengths of the sides.
- Repeat the steps for several other sets of consecutive integers.
- State your conjectures.

Drawings & Data

Conjectures

Do three consecutive integers always create a triangle? Why?

Do consecutive integers that do create triangles, create the same type of triangle? Why?

Conjectures

- No. The lengths 1–2–3 will not form a triangle.
- No. Lengths 2–3–4 form an <u>obtuse</u> triangle.

  Lengths 3–4–5 form a <u>right</u> triangle.

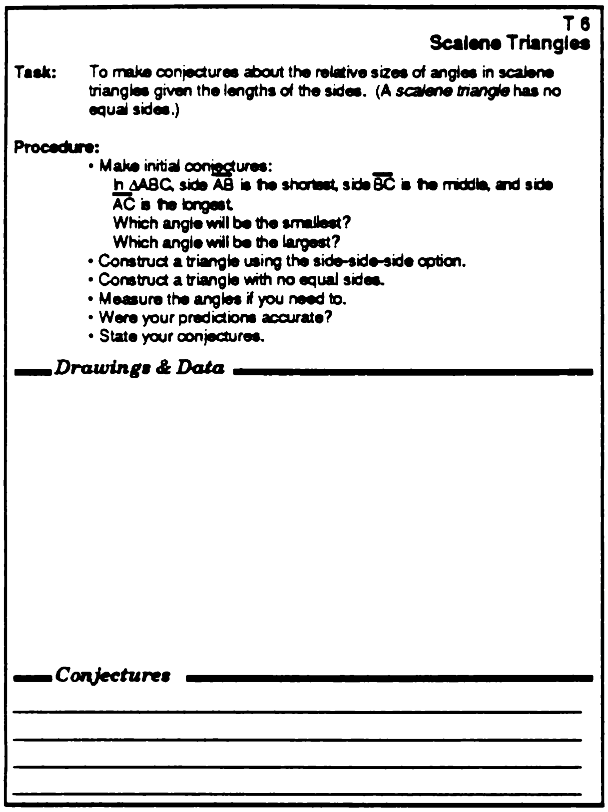
  Lengths n, $n + 1$, and $n + 2$ for $n \geq 4$ form an <u>acute</u> triangle.

Notes

After students understand the concept of similarity, ask them how to use the SUPPOSER to draw triangles whose sides are lengths 19–20–21 or 56–57–58. (Triangles whose sides have lengths 1.9–2.0–2.1 or 5.6–5.7–5.8 are similar to triangles with lengths 19–20–21 or 56–57–58.)

Here is another question to try: If triangles have sides 1000–1001–1002; 10,001–10,002–10,003; or 100,001–100,002–100,003; what type of triangle is it "approaching?" (Answer: An equilateral triangle)

Comments


T 6
Scalene Triangles

Task: To make conjectures about the relative sizes of angles in scalene triangles given the lengths of the sides. (A *scalene triangle* has no equal sides.)

Procedure:
- Make initial conjectures:
  In $\triangle ABC$, side $\overline{AB}$ is the shortest, side $\overline{BC}$ is the middle, and side $\overline{AC}$ is the longest.
  Which angle will be the smallest?
  Which angle will be the largest?
- Construct a triangle using the side-side-side option.
- Construct a triangle with no equal sides.
- Measure the angles if you need to.
- Were your predictions accurate?
- State your conjectures.

Drawings & Data

Conjectures

Conjectures

In response to this problem, students are likely to develop different versions of the same rule. Here are three possibilities:

- Order the lengths from shortest to longest. The smallest angle will be opposite the shortest side, the medium size angle will be opposite the side of medium length, and the largest angle will be opposite the longest side.

- The relationship of the sizes of the angles will be the same as the relationship of the lengths of their opposite sides.

- The smallest angle will be opposite the shortest side; the largest angle will be opposite the longest side.

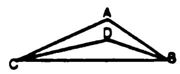
Comments

T 7 and T 8
Angles in Triangles
Angles Formed by Subdividing Triangles
Teacher Notes

T 7
Angles in Triangles

Task: To explore the measure of a vertex angle, created from a random point inside a triangle.

Procedure:
- Construct a $\triangle ABC$.
- Label a random point D inside $\triangle ABC$ and draw $\overline{BD}$ and $\overline{CD}$.



- Measure $\angle BDC$ and $\angle BAC$.
- Record the drawing and angle measurements.
- Repeat this procedure (do NOT use the Repeat option) on four other triangles.
- State your conjectures.

Triangle Drawings	$\angle BDC$	$\angle BAC$
1.		
2.		
3.		
4.		
5.		

Conjectures

T 8
Angles Formed by Subdividing Triangles

Task: To explore angles formed by subdividing a triangle.

Procedure:
- Construct a $\triangle ABC$. Measure $\angle BAC$.
- Place a random point D on $\overline{BC}$ of $\triangle ABC$.
- Draw $\overline{AD}$. Subdivide $\overline{AD}$ into four parts, using the Label option.
- Measure and draw $\angle BEC$, $\angle BFC$, and $\angle BGC$.
- Record the drawing and the measurements of the angles.
- Repeat the procedure with other triangles.
- State your conjectures.

Triangle Drawings	$\angle BAC$	$\angle BEC$	$\angle BFC$	$\angle BGC$
1.				
2.				
3.				
4.				
5.				

Conjectures

State a conjecture about the relationship between $\angle BAC$ and the other angles.

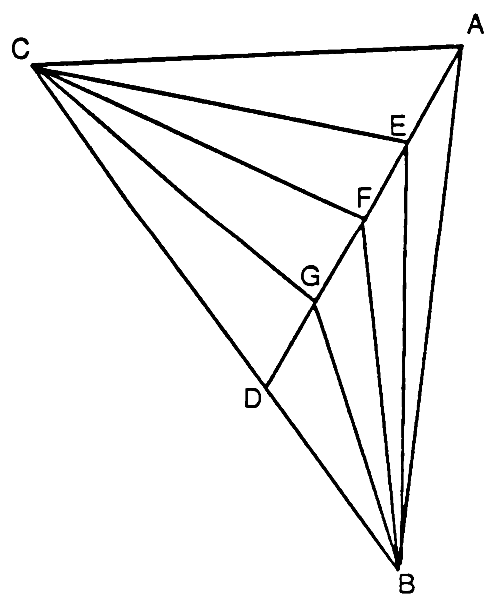
Conjectures

T 7

- $\angle BDC > \angle BAC$
- The closer point D is to point A, the closer the measure of the angles.

T 8

- The closer the angles are to $\angle BAC$, the closer they are to the measure of $\angle BAC$.



Comments

Angle-Side-Angle

Teacher Notes

T 9
Angle-Side-Angle

Task: To investigate the measurements that produce triangles using angle-side-angle.

Procedure:
- Use the measurements listed in the chart below.
- Construct a triangle using the angle-side-angle option.
- Record the drawing of each triangle on the chart.
- State your conjectures on the following page.

Triangle Drawings	∠BAC	AB	∠CBA
1.	100	1	45
2.	100	1	55
3.	100	1	65
4.	100	0.5	75
5.	80	3	40
6.	80	3	60
7.	80	3	80
8.	80	3	100
9.			

T 9 (page 2)
Angle-Side-Angle

Conjectures

In order to make a triangle, are there any constraints or limits that should be placed on the measures of ∠BAC, ∠CBA, or $\overline{AB}$?

Conjectures

- There is a limit to the sum of the measurement of ∠BAC and ∠CBA. Once the sum exceeds this limit, a triangle cannot be drawn.
- If ∠BAC + ∠CBA < 180°, then a triangle is formed.
- If ∠BAC + ∠CBA = 180°, then two sides will be parallel and a triangle cannot be formed.
- If ∠BAC + ∠CBA > 180°, then a triangle cannot be drawn.

Comments

T 10
Side-Angle-Side

Task: To investigate the measurements that produce triangles using side-angle-side.

Procedure:
- Use the measurements listed in the chart below.
- Construct a triangle using the side-angle-side option.
- Record the drawing of each triangle on the chart.
- State your conjectures on the following page.

Triangle Drawings	AB	∠BAC	AC
1.	7	20	4
2.	7	40	4
3.	7	60	4
4.	7	90	4
5.	7	110	4
6.	5	100	5
7.	5	90	5
8.	5	80	5
9.	5	20	5

T 10 (page 2)
Side-Angle-Side

Conjectures

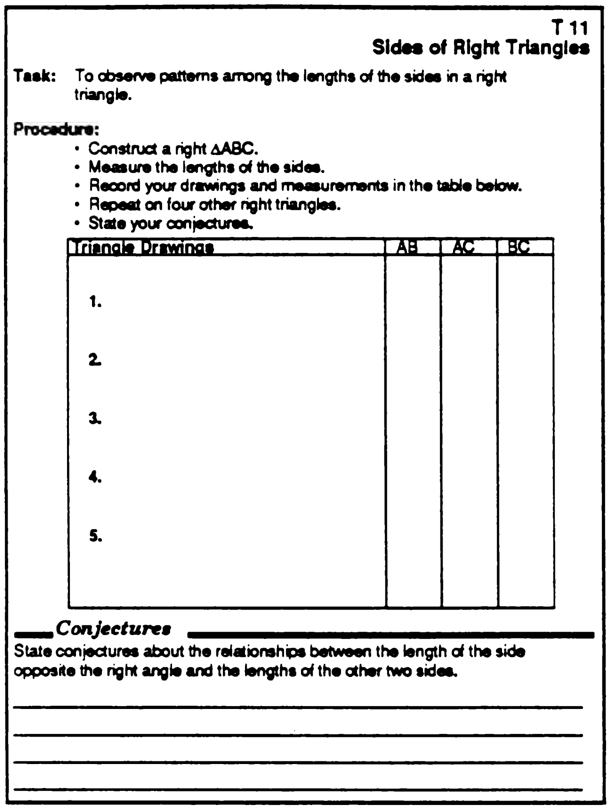
In order to make a triangle, are there any constraints or limits that should be placed on the measures of $\overline{AB}$, ∠BAC, or $\overline{AC}$?

Conjectures

- No constraints or limits.

Comments

T 11
Sides of Right Triangles
Teacher Notes



T 11
Sides of Right Triangles

Task: To observe patterns among the lengths of the sides in a right triangle.

Procedure:
- Construct a right $\triangle ABC$.
- Measure the lengths of the sides.
- Record your drawings and measurements in the table below.
- Repeat on four other right triangles.
- State your conjectures.

Triangle Drawings	AB	AC	BC
1.			
2.			
3.			
4.			
5.			

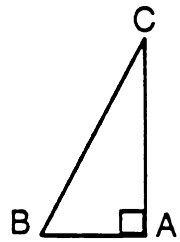
Conjectures

State conjectures about the relationships between the length of the side opposite the right angle and the lengths of the other two sides.

Conjectures

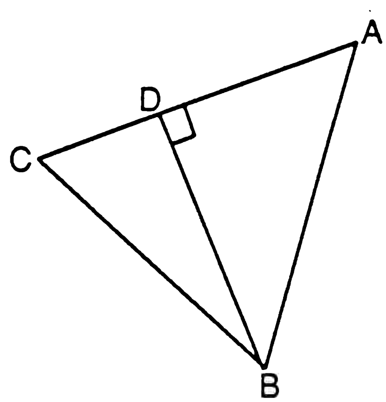
- In a right triangle, the length of the hypotenuse is greater than the length of either of the other two sides.

- Pythagorean Theorem



$$AC^2 + AB^2 = BC^2$$

Extension



This result can be used to develop a proof of relationships in the lengths of the sides of other types of triangles (See Problem T 12). For example, draw an acute triangle and an altitude $\overline{BD}$ from vertex B. Then, $AB > AD$ and $BC > DC$ implies that $AB + BC > AD + DC$ or $AB + BC > AC$. A similar method can be used to show that $AB + AC > BC$ and $AC + BC > AB$.

Comments

T 12
Side-Side-Side

Task: To investigate the measurements that produce triangles using side-side-side.

Procedure:
- Construct a ΔABC using the side-side-side option.
- If a triangle is formed, state what type it is.
- Record the lengths and the squares of the lengths in the table below.
- Repeat for three other triangles.
- State your conjectures.

Triangle Drawing	Lengths AB AC BC	Squares of Lengths $(AB)^2$ $(AC)^2$ $(BC)^2$	Forms a Triangle?	Type of Triangle

Conjectures

State conjectures about which combinations make triangles, which do not, and what type of triangle is formed.

Conjectures

- The sum of the lengths of any two sides is greater than the third side.
- If the sum of two lengths is less than the third length, then a triangle cannot be formed.
- The ratio of the sum of the lengths of any two sides to the length of the third side is greater than 1.
- If the difference between two lengths is greater than a third length, then a triangle cannot be formed.
- If the sum of the squares of the lengths of any two sides is greater than the square of the length of the third side, then the triangle is acute.
- If the sum of the squares of the lengths of any two sides is less than the square of the length of the third side, then the triangle is obtuse.
- Pythagorean Theorem

Notes

Let students develop their own procedures to test whether three lengths form a triangle. Some students may test a series of three lengths at random. Others may develop a system such as holding two lengths constant and varying the third length.

Comments

Supporting Conjectures

Teacher Notes

T 13

Supporting Conjectures

Task: To collect data to support conjectures.
To evaluate conditions under which a statement is true.

Procedure:

- The conjectures below are **always true**(A), **sometimes true**(S), or **never true**(N).
- Read each conjecture and place the appropriate letter (A, S, or N) next to each statement.
- Justify your answer using one of the following procedures:
  If you think a conjecture is **always true** (A), provide three examples.
  If you think a conjecture is **sometimes true** (S), provide two examples (one true and one false).
  If you think that a conjecture is **never true** (N), provide a counter-example.
- Be sure that your examples contain the appropriate measurements.

Conjectures

____ 1. A scalene triangle is not a right triangle.

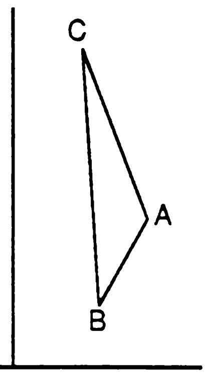
____ 2. An isosceles triangle is not a right triangle.

____ 3. An exterior angle of a triangle has a measure greater than the measure of any interior angle.

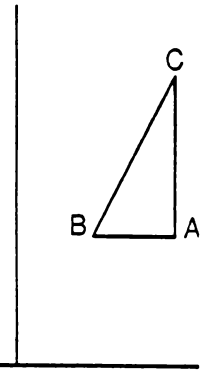
Conjectures

S 1.

Data
AB = 3.33
BC = 8.31
AC = 5.77

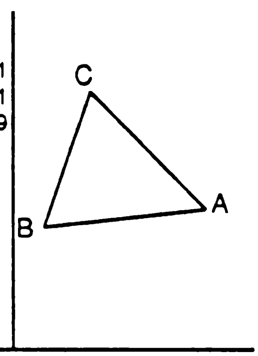


Data
AB = 2.76
BC = 5.71
AC = 5
<CAB=90

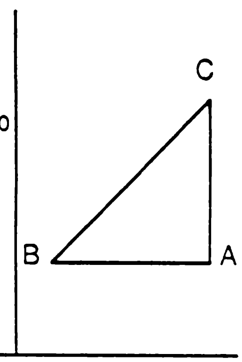


S 2.

Data
<ABC = 66.71
<BCA = 66.71
<CAB = 46.59
AB = 5.43
AC = 5.43
AC = 5

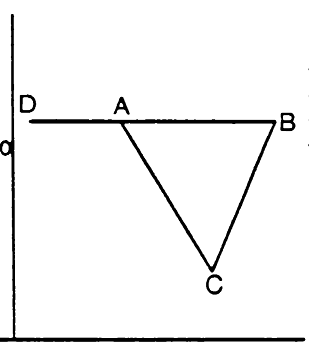


Data
AB = 6
AC = 6
<BAC = 90

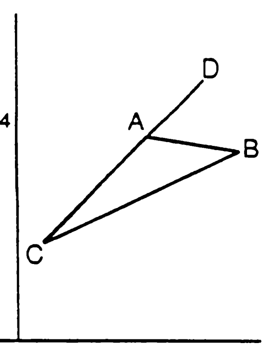


S 3.

Data
<ABC = 60
<BCA = 60
<CAB = 60
<CAD = 120



Data
<ABC =32.87
<BCA = 21.28
<CAB = 125.84
<BAD = 54.16



Comments

T 13 (page 2)
Supporting Conjectures

_____ 4. An exterior angle of a triangle has a measure greater than the measure of any remote interior angle.

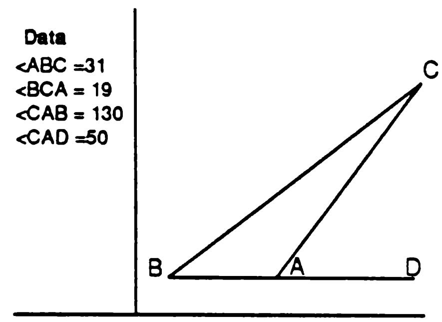
_____ 5. The longest side of a right triangle is the side opposite the right angle.

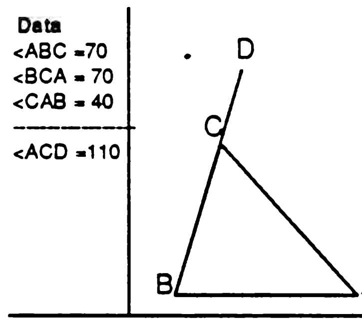
_____ 6. In any triangle, a median drawn from a vertex to a side bisects that side.

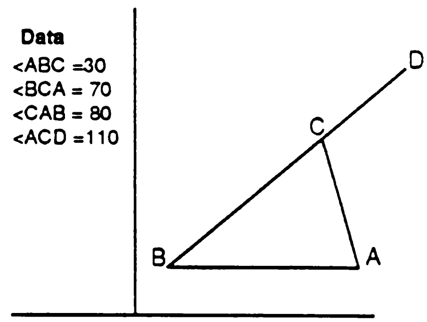
_____ 7. In any triangle, a median drawn from a vertex to a side bisects the angle at the vertex.

Conjectures

A 4.



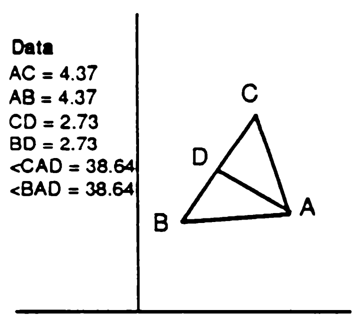


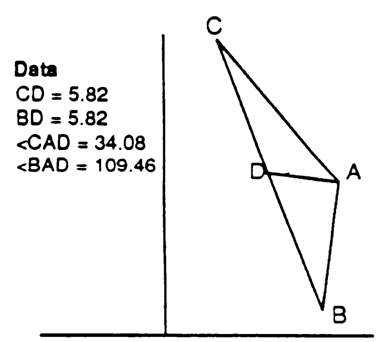


A 5. Any three right triangles.

A 6. Any three triangles with measurements.

S 7.





(Special thanks to Ruth Byron of Bedford High School in Bedford, MA for this problem.)

Comments

Area of a Triangle

Teacher Notes

T 14

Area of a Triangle

Task: To understand this formula for computing the area of a triangle:

Area $= \frac{1}{2}$ x (measure of the base) x (measure of the altitude) or $A = \frac{bh}{2}$

(Definition: Any side of a triangle can be called a *base*. Given a base, a line segment drawn from the remaining vertex, perpendicular to the base or the extension of the base, is called an *altitude* (or *height*) of the triangle.)

Procedure:

- Construct an isosceles ΔABC. Draw an altitude from vertex A.
- Measure the altitude and the base.
- Calculate the value of bh/2, using a calculator.
- Record your data in the chart.
- Repeat this procedure by drawing the altitude from B. Then do the same for vertex C.
- Check your calculations using the Area option to measure the area.
- Repeat this procedure for an acute triangle. Then repeat for an obtuse triangle.
- State your conjectures on the following page.

Triangle Drawings	Base	Altitude	Area (by formula)	Area (by Area option)
Isosceles Triangle				
Acute Triangle				
Obtuse Triangle				

T 14 (page 2)

Area of a Triangle

Conjectures

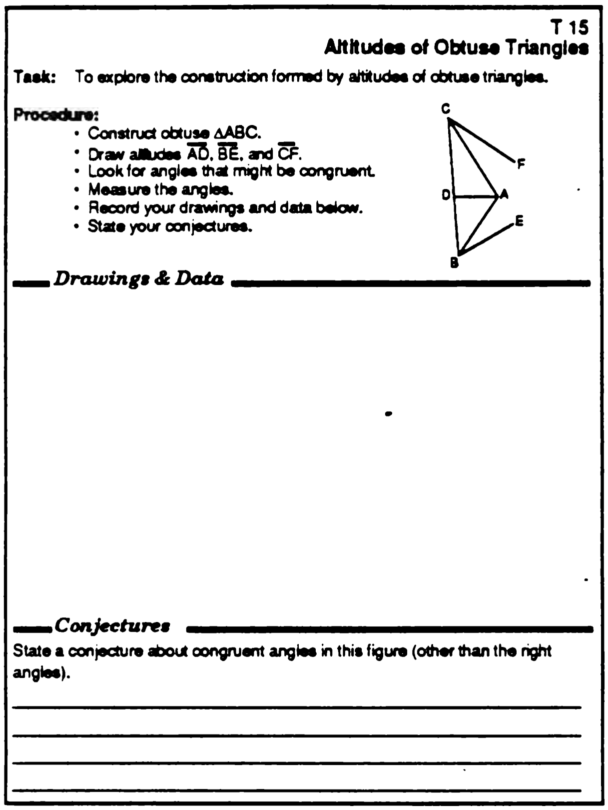
Notes

The purpose of these activities is to convince students that the area of a triangle can be found using the <u>same</u> formula with <u>three different pairs</u> of bases and altitudes, and that the formula will work even if the altitudes lie outside the triangle (the case with an obtuse triangle).

Comments

Altitudes of Obtuse Triangles

Teacher Notes


T 15
Altitudes of Obtuse Triangles

Task: To explore the construction formed by altitudes of obtuse triangles.

Procedure:
- Construct obtuse $\triangle ABC$.
- Draw altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.
- Look for angles that might be congruent.
- Measure the angles.
- Record your drawings and data below.
- State your conjectures.

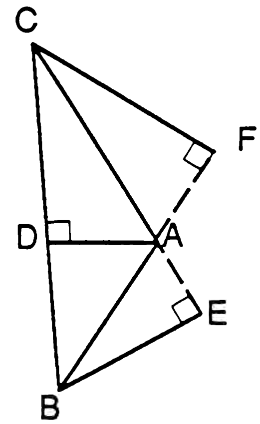
Drawings & Data

Conjectures

State a conjecture about congruent angles in this figure (other than the right angles).

Conjectures

- $\angle BAE = \angle CAF$
- $\angle BEA = \angle CFA = 90°$.



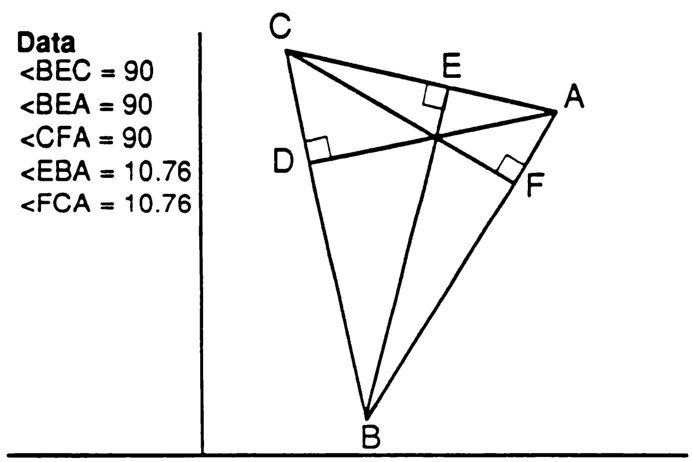
- $\angle EBA = \angle FCA$

Proof

$\angle EBA = \angle FCA$ since the sum of the angles in any triangle is 180°, the measures of vertical angles ($\angle FAC$ and $\angle BAE$) are equal, and $\angle BEA = \angle CFA = 90°$.

Notes

Check this conjecture with different types of triangles. For example, in an acute $\triangle ABC$:



Comments

Triangles from Altitudes

Teacher Notes

T 16

Triangles from Altitudes

Task: To investigate the triangle formed by joining the endpoints of the three altitudes.

Procedure:

- Construct an acute $\triangle ABC$.
- Draw an altitude from each vertex.
- Erase the altitudes, leaving the labels D, E, and F.
- Now draw $\triangle DEF$.
- Determine whether the $\triangle DEF$ extends beyond the original $\triangle ABC$.
- Check whether $\triangle DEF$ is a right triangle.
- Repeat on several other triangles.

Drawings & Data

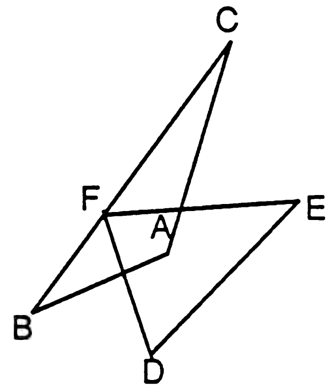
Conjectures

In what kind of triangle does the region defined by $\triangle DEF$ extend beyond the exterior of $\triangle ABC$?

Can $\triangle DEF$ ever be a right triangle? ____________

Suppose $\triangle ABC$ is an isosceles triangle with vertex $\angle A$. When the vertex angle of $\triangle ABC$ is close to 90°, the perimeter of $\triangle DEF$ will be close to what measure? ____________

Conjectures



- The region defined by $\triangle DEF$ extends beyond $\triangle ABC$ when $\triangle ABC$ is **obtuse**.
- Yes, $\triangle DEF$ is a right triangle when $\angle BAC = 135°$. You could also try a special acute triangle with 45°, 65°, and 70° angles.
- When the vertex angle of an isosceles $\triangle ABC$ approaches 90°, the perimeter of $\triangle DEF$ approaches the length of the hypotenuse of the right isosceles $\triangle ABC$.

Comments

T 17
Angle Bisectors
Teacher Notes

T 17
Angle Bisectors

Task: To understand an angle bisector as a set of points that share a certain property.

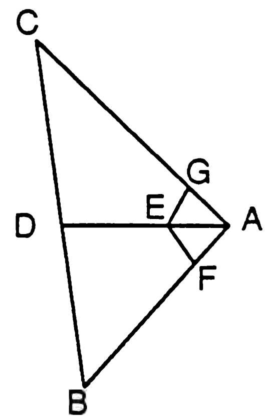
Procedure:
- Construct a right $\triangle ABC$.
- Draw angle bisector $\overline{AD}$.
- Place a random point E on the bisector.
- Measure the distance from E to $\overline{AB}$ and the distance from E to $\overline{AC}$.
- Record your data and the drawing.
- Repeat for each triangle listed. Do NOT use the Repeat option.
- State your conjectures on the next page.

Triangle Drawings		Distance from E to $\overline{AB}$	Distance from E to $\overline{AC}$
Right			
Acute			
Obtuse			
Isosceles			
Equilateral			
Scalene			

T 17 (page 2)
Angle Bisectors

Conjectures

Conjectures



- The distance from E to $\overline{AB}$ equals the distance from E to $\overline{AC}$.
- AF = AG.
- The perimeter of $\triangle AFE$ = the perimeter of $\triangle AGE$.

Comments

T 18
Medians

Task: To understand a median as a set of points that share a certain property.

Procedure:
- Draw a $\triangle ABC$.
- Draw median $\overline{AD}$. Label a random point E on median $\overline{AD}$.
- Draw $\overline{CE}$ and $\overline{BE}$.
- Measure the areas of the triangles inside $\triangle ABC$.
- Record your data.
- State your conjectures.

Drawings & Data

Conjectures

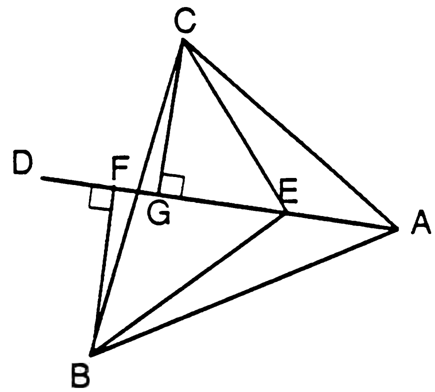
State conjectures about the relationships among the areas of the triangles inside $\triangle ABC$.

Conjectures

- Area of $\triangle CED$ = Area of $\triangle BED$.
- Area of $\triangle ACD$ = Area of $\triangle ABD$.
- Area of $\triangle CEA$ = Area of $\triangle BEA$.

Proof

Prove: Area of $\triangle CEA$ = Area of $\triangle BEA$



Draw $\overline{BF} \perp \overline{AD}$ and $\overline{CG} \perp \overline{AD}$
Thus, $\overline{BF} \parallel \overline{GC}$. Since $\angle FBD \cong \angle GCD$ (alternate interior angles) and $\overline{BD} \cong \overline{CD}$, $\triangle FBD \cong \triangle GCD$ by ASA and $\overline{BF} \cong \overline{CG}$ by corresponding parts of congruent triangles.

Now, the area of $\triangle AEB = \dfrac{AE \cdot BF}{2}$ and the area of $\triangle AEC = \dfrac{AE \cdot CG}{2}$.

The areas are equal because BF = CG.

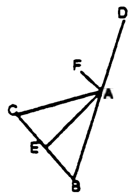
Comments

T 19 and T 20
Angle Bisectors at a Vertex
Angle Bisectors of Exterior Angles
Teacher Notes

T 19
Angle Bisectors at a Vertex

Task: To explore the angle formed by the angle bisector of the exterior and interior angles at a vertex.

Procedure:
- Construct acute $\triangle ABC$.
- Extend $\overline{AB}$ from A.
- Draw $\overline{AE}$, the bisector of $\angle BAC$.
- Draw $\overline{AF}$, the bisector of $\angle CAD$.
- Measure $\angle FAE$.
- State your conjectures.



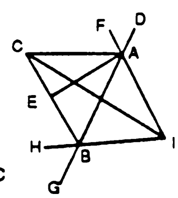
Drawings & Data

Conjectures

T 20
Angle Bisectors of Exterior Angles

Task: To explore properties of the intersection point of the angle bisectors of two exterior angles.

Procedure:
- Construct acute $\triangle ABC$.
- Extend $\overline{AB}$ from A.
- Draw $\overline{AE}$, the bisector of $\angle BAC$.
- Draw $\overline{AE}$, the bisector of $\angle CAD$.
- Extend $\overline{AB}$ from B.
- Draw $\overline{BH}$, the bisector of $\angle GBC$.
- Label the intersection of the two bisectors ($\overline{FA}$ and $\overline{HB}$) of the exterior angles for $\triangle ABC$ with point I.
- Draw $\overline{IC}$.
- State conjectures related to segment $\overline{IC}$.



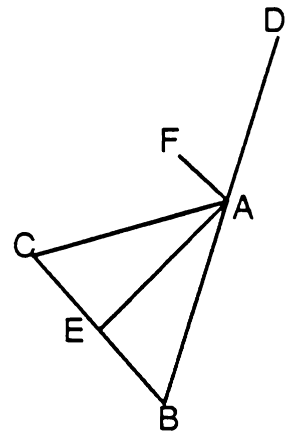
Drawings & Data

Conjectures

Conjectures

T 19

- $\angle FAE = 90°$



Proof

prove: $\angle FAE = 90°$
$\angle CAF = \angle DAF$
$\angle BAE = \angle CAE$
$2(\angle CAE) + 2(\angle CAF) = 180°$
$\angle CAE + \angle CAF = 90°$
So $\angle FAE = 90°$

Conjectures

T 20

- $\overline{IC}$ is the bisector of $\angle ACB$, $\angle ACI = \angle BCI$

- The point of intersection of the angle bisectors for $\triangle ABC$ lies on $\overline{IC}$.

Comments

Angle Bisectors and Side Lengths

Teacher Notes

T 21

Angle Bisectors and Side Lengths

Task: To explore the way an angle bisector divides the opposite side.

Procedure:

- Use the measurements from the chart below.
- Construct a $\triangle ABC$ using the side-angle-side option.
- Draw $\overline{AD}$, the bisector of $\angle BAC$.
- Observe the relative lengths of the segments $\overline{BD}$, $\overline{CD}$, $\overline{AB}$, and $\overline{AC}$.
- Record the drawings.
- State your conjectures.

Triangle Drawings	AB	∠CBA	AC
1.	8	100	2
2.	8	100	3
3.	8	100	4
4.	8	100	5
5.	8	100	6
6.	8	100	8

Conjectures

State conjectures about the relationships among $\overline{BD}$, $\overline{CD}$, $\overline{AB}$, and $\overline{AC}$, relative to the location of point D on $\overline{CB}$.

Conjectures

- If AB < AC, then BD < CD
- If AB > AC, then BD > CD
- If AB = AC, then BD = CD
- If AB < AC, then D is "closer" to B
- If AB > AC, then D is "closer" to C

Notes

This activity asks the student to attend to <u>visual data</u>, to the movement of point D (*i.e.*, the endpoint of the angle bisector) as the relationship between the lengths of the sides that enclose the bisected angle changes. Encourage students to check their conjectures through visual approximation rather than measurement.

Comments

T 22
Triangles Formed by an Angle Bisector
Teacher Notes

T 22
Triangles Formed by an Angle Bisector

Task: To explore the relationships between the two triangles formed when an angle bisector is drawn in a triangle.

Procedure:

- Construct an isosceles ΔABC.
- Draw $\overline{AD}$ bisecting $\angle CAB$.
- Measure the lengths, areas, and perimeters.
- Record your data.
- Repeat this procedure on other types of triangles.
- State your conjectures.

Triangle Type	LENGTHS AB	AC	BD	DC	Area ΔABD	Area ΔACD	Perimeter ΔABD	Perimeter ΔACD
Isosceles								
Acute								
Right								
Equilateral								
Obtuse								

Conjectures

Conjectures

- If ΔABC is isosceles and $\angle A$ is the vertex angle, then the area of ΔABD will equal the area of ΔACD and the perimeter of ΔABD will equal the perimeter of ΔACD.

- If ΔABC is isosceles and $\angle A$ is the vertex angle, then $BD = CD$.

- The same conjectures hold for equilateral triangles.

- $\dfrac{\text{Area of } \Delta ABD}{\text{Area of } \Delta ACD} = \dfrac{AB}{AC}$ in any triangle

- $\dfrac{AB}{AC} = \dfrac{BD}{DC}$ in any triangle

Comments

T 23
Angle Bisectors in Scalene Triangles
Teacher Notes

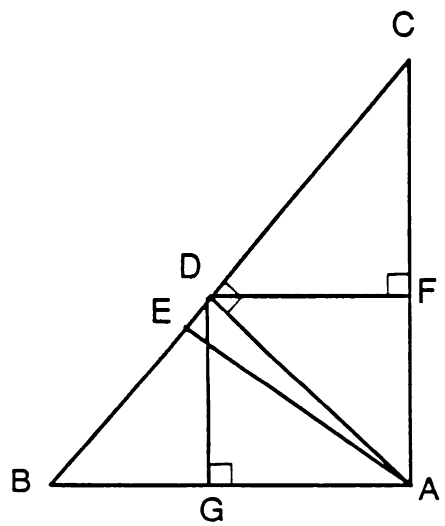
T 23
Angle Bisectors in Scalene Triangles

Task: To discover proportions among the elements and triangles in a scalene triangle with one angle bisector.

Procedure:
- Construct a scalene $\triangle ABC$.
- Draw the angle bisector of angle A.
- Measure the segments and triangles formed.
- Record your data.
- State your conjectures about relationships among the elements of the figure.

Drawings & Data

Conjectures



Conjectures

$$\frac{BD}{BA} = \frac{DC}{AC} \quad \text{or} \quad \frac{BA}{BD} = \frac{AC}{DC} \quad \text{or} \quad \frac{DC}{AC} \times \frac{BA}{BD} = 1$$

$$\text{or} \quad \frac{BA}{AC} = \frac{BD}{DC}$$

$$\frac{\text{Area of } \Delta BAD}{\text{Area of } \Delta CAD} = \frac{BD}{DC} = \frac{BA}{CA}$$

Proof

Note: The traditional way to prove this result is to use similarity. A less traditional proof (which does not use similarity theorems) is to use two different methods to compute the area of the same triangle:

Given: $\overline{AD}$ an angle bisector

$\overline{AE}$ a perpendicular to $\overline{BC}$ through A.

$\overline{DF}$ a perpendicular to $\overline{CA}$ through D.

$\overline{DG}$ a perpendicular to $\overline{AB}$ through D.

Prove: $\dfrac{CD}{BD} = \dfrac{AC}{AB}$

Proof: $\text{Area of } \Delta\, BAD = \dfrac{DB \cdot AE}{2} \quad \text{or} \quad \dfrac{AB \cdot DG}{2}$

$\text{Area of } \Delta\, CAD = \dfrac{CD \cdot AE}{2} \quad \text{or} \quad \dfrac{AC \cdot DF}{2}$

By dividing the two equations we get:

$$\frac{\dfrac{CD \cdot AE}{2}}{\dfrac{DB \cdot AE}{2}} = \frac{\dfrac{AC \cdot DF}{2}}{\dfrac{AB\ DG}{2}}$$

$$\frac{CD \cdot \cancel{AE}}{DB \cdot \cancel{AE}} = \frac{AC \cdot DF}{AB \cdot DG}$$

$$\frac{CD}{DB} = \frac{AC}{AB}$$

since DF = DG (D is on angle bisector $\overline{AD}$)

Comments

T 24
The Triangles Created by Medians
Teacher Notes

T 24
The Triangles Created by Medians

Task: To explore the relationships among the elements (sides, segments, angles) and the properties (area, perimeter) of the three triangles (△ABC, △ABD, △CAD) created when a median is drawn in a triangle.

Procedure:
- Construct a right △ABC.
- Draw the median $\overline{AD}$.
- Measure elements, area, and perimeter.
- Repeat on other triangles.
- State your conjectures on the following page.

Drawings & Data

Conjectures

Right Triangles:
- The median drawn from the vertex of the right angle is half the length of the hypotenuse.
- A median divides the triangle into two sections of equal area.

Acute Triangles:
- A median divides the triangle into two sections of equal area.

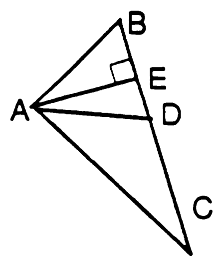
Obtuse Triangles:
- A median divides the triangle into two sections of equal area.

Isosceles Triangles:
- A median divides the triangle into two sections of equal area.
- A median from the vertex angle creates two triangles with equal perimeters.
- A median from the vertex angle can also be an **angle bisector**, **altitude**, or **perpendicular bisector**.

Equilateral Triangles:
- A median from any vertex can be an angle bisector, altitude, or perpendicular bisector.

Notes

In addition to exploring the properties of medians, this activity has a second purpose. Beginning geometry students often believe that if a relationship is true for one type of triangle, then it is true for all triangles. For example, a median drawn from the 90° angle in an **isosceles** right triangle is perpendicular to the hypotenuse. Students commonly think that this holds for medians in all right triangles. This is clearly false as the drawing shows.



Therefore, in this activity, students are asked to search, test, and state their conjectures in terms of specific types of triangles. They are not asked to make generalizations that apply to all triangles.

Comments

T 24 (page 2)
The Triangles Created by Medians
Teacher Notes

T 24 (page 2)
The Triangles Created by Medians

Conjectures

Right Triangles (non-isosceles):

____

____

Acute Triangles:

____

____

Obtuse Triangles:

____

____

Isosceles Triangles:

____

____

In an isosceles triangle, the median from the vertex angle can also be a __________ , a__________ , or a __________ .

Equilateral Triangles:

____

____

In an equilateral triangle, the median from any vertex of the triangle can also be a __________ , a __________ , or a __________ .

The table below can be used for class discussion to help students obtain a general view of conjectures they have collected for each type of triangle.

List your conjectures in the left hand column and put a "yes" or a "no" under the type of triangle to indicate whether a given conjecture is true for that type of triangle.

For each conjecture that holds for <u>more than two types</u> of triangles, you might ask students to write a convincing argument to support that conjecture.

Conjectures:	Right	Acute	Obtuse	Isosceles	Equilateral

Comments

T 25
Two Medians

Task: To explore relationships in a triangle when two medians are drawn.

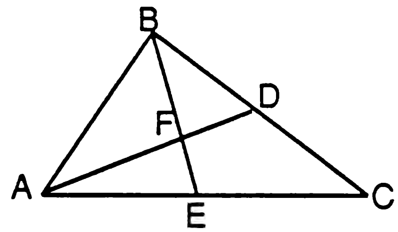
Procedure:
- Construct $\triangle ABC$.
- Draw medians $\overline{AD}$ and $\overline{BE}$.
- Label the intersection of the two medians with point F.
- State your conjectures.

Drawings & Data

Conjectures

____
____
____
____

Conjectures



- Area of $\triangle ABD$ = Area of $\triangle ABE$
- $\dfrac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ABF} = \dfrac{3}{2} = \dfrac{\text{Area of } \triangle ABE}{\text{Area of } \triangle ABF}$
- Area of $\triangle AEF$ = Area of $\triangle BFD$
- Area of $\triangle ABF$ = Area of $\triangle AEF$ + Area of $\triangle BFD$

Comments

T 26 and T 27
A Median and Sides
Three Medians

Teacher Notes

T 26
A Median and Sides

Task: To investigate the relationship between the length of the median and the lengths of the sides.

Procedure:

- Construct any $\triangle ABC$.
- Draw median $\overline{AD}$.
- Measure the length of the median and the lengths of the sides of the triangle.
- State your conjectures about these measurements.
- Investigate whether your conjectures hold true for different types of triangles.

Drawings & Data

Conjectures

T 27
Three Medians

Task: To investigate the relative lengths of medians in a triangle.

Procedure:

- Construct any $\triangle ABC$.
- Draw the three medians.
- Measure the lengths of the medians and the lengths of the sides of the triangle.
- State your conjectures.

Drawings & Data

Conjectures

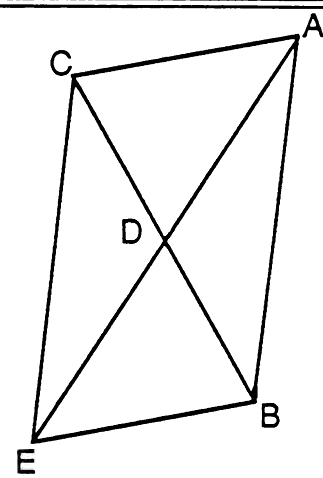
In relation to the lengths of the sides, which median do you think will be the shortest? Which median do you think will be the longest?

Conjectures

T 26

- $AD < \frac{AB + AC}{2}$

Proof



Extend $\overline{AD}$ such that ED = AD.
AB = CE (ABEC is a parallelogram)
In $\triangle ACE$, $AC + EC > AE$ implies $AB + AC > AE = 2 \cdot AD$

and $AD < \frac{AB + AC}{2}$.

Conjectures

T 27

- The shortest median will be drawn to the longest side.
- The longest median will be drawn to the shortest side.

Comments

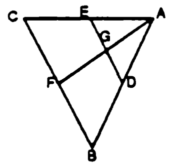
T 28
One Midsegment
Teacher Notes

T 28
One Midsegment

Task: To explore figures formed by drawing one midsegment in a triangle.

Procedure:

- Construct any $\triangle ABC$.
- Draw midsegment $\overline{DE}$ connecting $\overline{AB}$ and $\overline{AC}$.
- Label the midpoint of $\overline{BC}$ with point F.
- Draw $\overline{AF}$ and label the intersection of $\overline{DE}$ and $\overline{AF}$ with point G.
- Measure the elements of the figure.
- Record your data.
- State your conjectures.



Drawings & Data

Conjectures

Conjectures

- $\overline{ED} \parallel \overline{BC}$
- $ED = \frac{1}{2} \cdot BC$
- $EG = DG$, $AG = GF$

- Area of ΔEAG = Area of ΔDAG
- Area of ΔACF = Area of ΔABF
- Area of EGFC = Area of DGFB

- $\angle AEG = \angle ACF$, $\angle ADG = \angle ABF$
  $\angle EGA = \angle CFA$, $\angle AGD = \angle AFB$
  $\Delta AED \sim \Delta ACB$

- $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ADE} = 4 \qquad \dfrac{\text{Area of } \Delta ACF}{\text{Area of } \Delta AEG} = 4$

- $\dfrac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta ADE} = 2 \qquad \dfrac{\text{Perimeter of } \Delta ACF}{\text{Perimeter of } \Delta AEG} = 2$

Comments

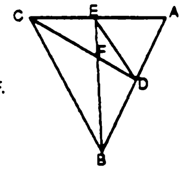
One Midsegment and Two Medians

Teacher Notes

T 29
One Midsegment and Two Medians

Task: To explore figures formed by a midsegment and two medians.

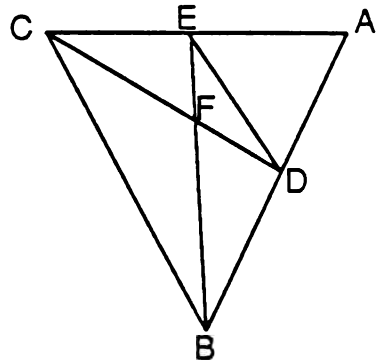
Procedure:
- Construct any $\triangle ABC$.
- Draw midsegment $\overline{DE}$ connecting $\overline{AB}$ and $\overline{AC}$.
- Draw $\overline{BE}$ and $\overline{CD}$.
- Label the intersection of $\overline{BE}$ and $\overline{CD}$ with point F.
- Measure the elements of the figure.
- Record your data.
- State your conjectures.



Drawings & Data

Conjectures

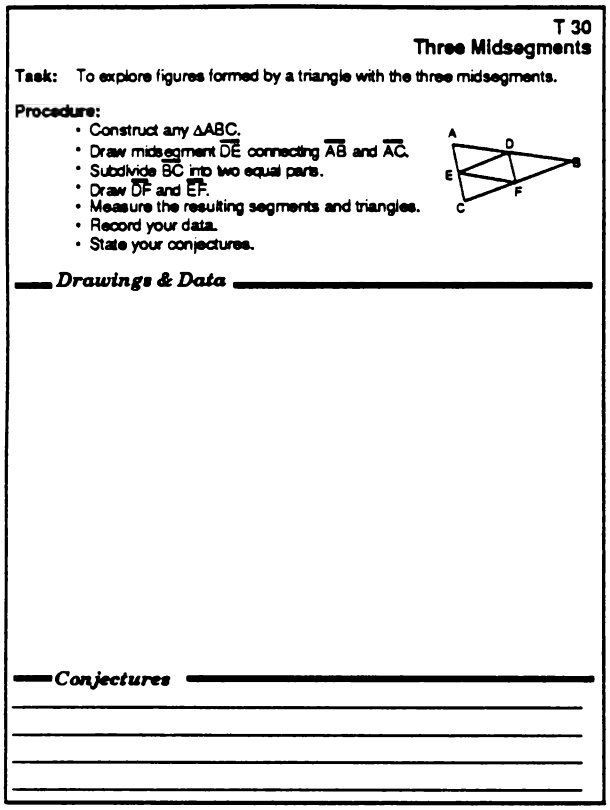
Conjectures



- $\dfrac{\text{Area of } \triangle ABC}{\text{Area of } \triangle AED} = 4$
- $\dfrac{\text{Area of } \triangle CFB}{\text{Area of } \triangle EFD} = 4$
- $\dfrac{\text{Area of } \triangle CFB}{\text{Area of } \triangle AED} = \dfrac{4}{3}$
- Area of $\triangle CEF$ = Area of $\triangle DFB$
- $\dfrac{\text{Perimeter of } \triangle ABC}{\text{Perimeter of } \triangle AED} = 2$
- $\dfrac{\text{Perimeter of } \triangle CFB}{\text{Perimeter of } \triangle EFD} = 2$
- $DE = \dfrac{1}{2} \cdot BC$
- $\overline{DE} \parallel \overline{BC}$

Comments

T 30
Three Midsegments
Teacher Notes


T 30
Three Midsegments

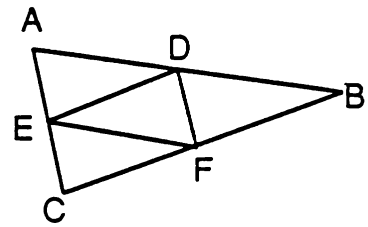
Task: To explore figures formed by a triangle with the three midsegments.

Procedure:
- Construct any ΔABC.
- Draw midsegment $\overline{DE}$ connecting $\overline{AB}$ and $\overline{AC}$.
- Subdivide $\overline{BC}$ into two equal parts.
- Draw $\overline{DF}$ and $\overline{EF}$.
- Measure the resulting segments and triangles.
- Record your data.
- State your conjectures.

Drawings & Data

Conjectures

Conjectures

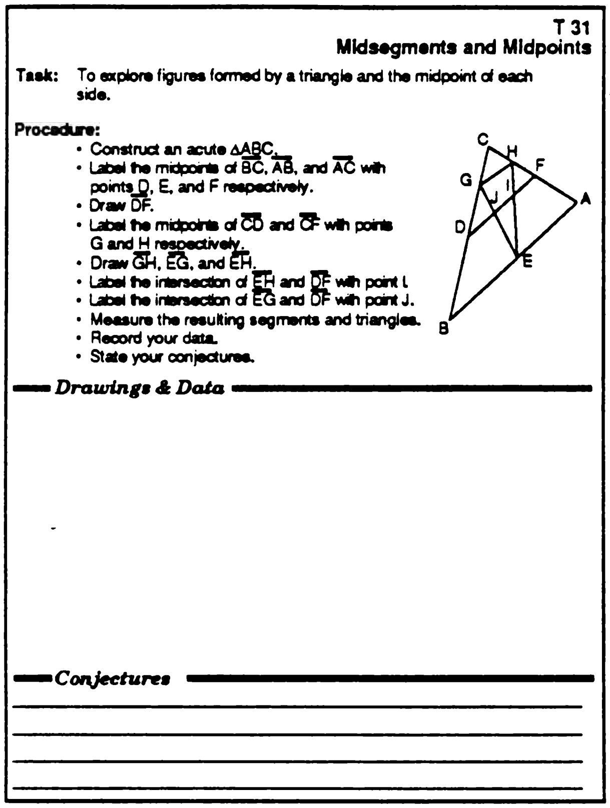


- $EF = \frac{1}{2} \cdot AB$, $DE = \frac{1}{2} \cdot BC$, $DF = \frac{1}{2} \cdot AC$
- $\overline{EF} \parallel \overline{AB}$, $\overline{DE} \parallel \overline{BC}$, $\overline{DF} \parallel \overline{AC}$.
- The four triangles inside ΔABC are congruent and have equal areas and perimeters.
- If ΔABC is an "x" triangle, then each of the four triangles inside ΔABC are "x" triangles.
- BDEF, DECF, ADFE are parallelograms.
- ABFE, ADFC, EDBC are trapezoids.

Comments

Midsegments and Midpoints

Teacher Notes



T 31
Midsegments and Midpoints

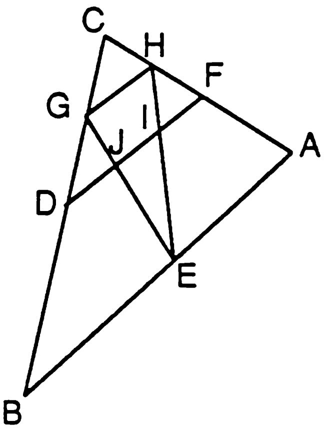
Task: To explore figures formed by a triangle and the midpoint of each side.

Procedure:
- Construct an acute $\triangle ABC$.
- Label the midpoints of $\overline{BC}$, $\overline{AB}$, and $\overline{AC}$ with points D, E, and F respectively.
- Draw $\overline{DF}$.
- Label the midpoints of $\overline{CD}$ and $\overline{CF}$ with points G and H respectively.
- Draw $\overline{GH}$, $\overline{EG}$, and $\overline{EH}$.
- Label the intersection of $\overline{EH}$ and $\overline{DF}$ with point I.
- Label the intersection of $\overline{EG}$ and $\overline{DF}$ with point J.
- Measure the resulting segments and triangles.
- Record your data.
- State your conjectures.

Drawings & Data

Conjectures

Conjectures

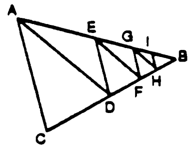


- Area of ΔFIH = Area of ΔDJG
- Areas of $\Delta FIH + \Delta DJG$ = Area of ΔEIJ
- $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta CHG} = 16$
- $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta CDF} = 4$
- $\dfrac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta CHG} = 4$
- Area of AFIE = Area of BEJD
- $GH = \dfrac{AB}{4} = \dfrac{DF}{2}$
- $FI = IJ = JD$

Comments

T 32
Parallel Lines

Task: To reproduce the figure below, **without** using the Parallel option. In ΔABC, $\overline{DE} \parallel \overline{FG} \parallel \overline{HI}$ and $\overline{AD} \parallel \overline{EF} \parallel \overline{GH}$.



Procedure:
- Make a drawing similar to the figure above.
- Collect data to confirm that the segments are parallel.
- Describe below the procedures for reproducing the figure.
- Develop three methods for drawing parallel lines without using the Parallel option.

Drawings & Data

Conjectures

Procedure for reproducing the figure:

Three methods for drawing parallel lines inside triangles **without** using the Parallel option. Describe your methods and record sample drawings for each method.

1.

2.

3.

Conjectures

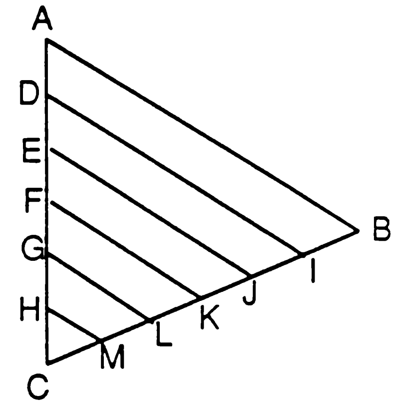
<u>Procedure for reproducing the figure:</u>

Draw median $\overline{AD}$ in ΔABC, median $\overline{DE}$ in ΔADB, median $\overline{EF}$ in ΔDEB, median $\overline{FG}$ in ΔEFB, median $\overline{GH}$ in ΔFGB, and median $\overline{HI}$ in ΔGHB.

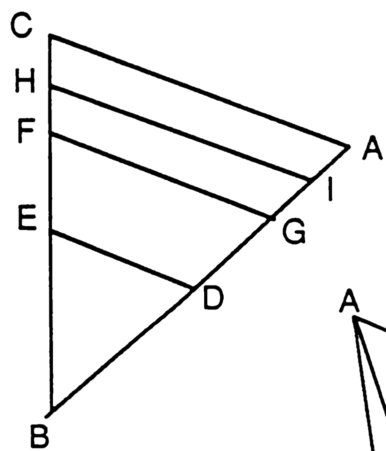
By measuring the angles, you can show that the alternate interior angles are equal. For example, $\overline{AD} \parallel \overline{EF}$ because $\angle ADE = \angle DEF$. The Midline or Midsegment Theorem can be used to prove the result.

<u>Three methods for drawing parallel lines:</u>

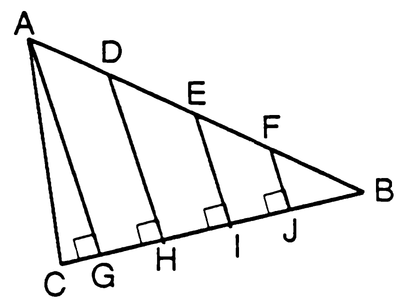
1. Subdivide sides $\overline{AC}$ and $\overline{BC}$ into an equal number (n) of subdivisions and draw the segments as in the diagram below. By measuring angles, you can show that the appropriate corresponding angles are equal.



2. Draw midsegment $\overline{DE}$ in ΔABC from $\overline{AB}$ to $\overline{BC}$ and draw midsegment $\overline{FG}$ in trapezoid ACED. Repeat this process.



3. In ΔABC, subdivide $\overline{AB}$ into n equal subdivisions. Draw perpendiculars from each point on $\overline{AB}$ to side $\overline{BC}$.



Comments

Parallelogram in a Triangle

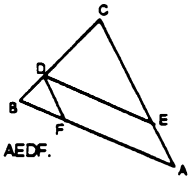
Teacher Notes

T 33

Parallelogram in a Triangle

Task: To investigate parallelograms drawn in isosceles triangles.

Procedure:



- Construct an isosceles $\triangle ABC$ using the side-angle-side option so that AC = AB.
- Label a random point D on $\overline{BC}$.
- Construct $\overline{DE}$ such that $\overline{DE}$ is parallel to $\overline{AB}$.
- Construct $\overline{DF}$ such that $\overline{DF}$ is parallel to $\overline{AC}$.
- Measure the sum of the lengths AB + AC.
- Measure the perimeter of the parallelogram AEDF.
- Record your data.
- Repeat this procedure for different types of triangles, using a new random point on each triangle.
- State your conjectures.

Drawings & Data

Conjectures

What is the relationship between AB + AC and the perimeter of parallelogram AEDF?

Conjectures

- AB + AC = perimeter of AEDF, if ΔABC is isosceles or equilateral.

Proof

After students have an understanding of the relationships between similar triangles, ask them to prove this relationship:

$\Delta CDE \sim \Delta CBA$ implies that ΔECD is an isosceles triangle with CE = DE. $\Delta BDF \sim \Delta BCA$ implies DF = BF.

$$\begin{aligned} \text{Thus, } AB + AC &= BF + FA + AE + CE \\ &= DF + FA + AE + DE \quad \text{(by substitution)} \\ &= \text{Perimeter of } AEDF \end{aligned}$$

Conjectures

- If ΔABC is a right triangle, then AEDF is a rectangle.
- $\Delta ABC \sim \Delta EDC \sim \Delta FBD$

Comments

Perpendiculars in a Triangle

Teacher Notes

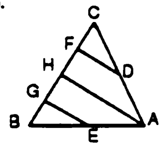
T34

Perpendiculars in a Triangle

Task: To explore the relationship between perpendiculars drawn from midpoints of two sides in a triangle to the third side.

Procedure:

- Construct any $\triangle ABC$.
- Label the midpoints of $\overline{AC}$ and $\overline{AB}$ with points D and E respectively.
- Draw $\overline{DF}$ and $\overline{EG}$ so that they are perpendicular to $\overline{BC}$.
- Investigate the relationship between DF and EG.
- Record your data.
- Draw the altitude $\overline{AH}$.
- Investigate the relationships among DF, EG, and altitude AH.
- State your conjectures.



Drawings & Data

Conjectures

Conjectures

- DF = EG
- $\overline{DF} \parallel \overline{EG}$
- DF + EG = AH
- BG + CF = GF
- $\dfrac{\text{Area of } \triangle ABC}{\text{Area of } \triangle CDF + \text{Area of } \triangle BGE} = 4$
- $\dfrac{\text{Area of } \triangle AHC}{\text{Area of } \triangle DFC} = \dfrac{\text{Area of } \triangle BAH}{\text{Area of } \triangle BEG} = 4$
- $\triangle AHC \sim \triangle DFC \sim \triangle EGB$

Comments

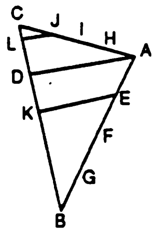
T 35
Lengths of Perpendiculars
Parts A and B
Teacher Notes

T 35

Lengths of Perpendiculars - Part A

Task: To investigate relationships among perpendicular line segments drawn from points on one side to another side.

Procedure:
- Study this figure.
- Repeat these constructions on other triangles.
- Measure line segments.
- Record your data.
- State your conjectures.



Drawings & Data

Conjectures

T 35

Lengths of Perpendiculars - Part B

Task: To investigate relationships among perpendicular line segments drawn from points on two sides to the third side.

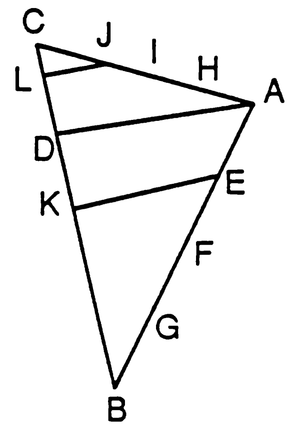
Procedure:
- Construct any $\triangle ABC$.
- Subdivide any two sides into three segments.
- Draw perpendiculars from the subdivisions to the third side.
- Repeat this drawing on different types of triangles.
- Repeat the same construction with four subdivisions and five subdivisions.
- Record your data in the chart below.
- State your conjecture.

Number of Subdivisions (n)	Sum of the lengths of the perpendiculars along $\overline{AB}$	Sum of the lengths of the perpendiculars along $\overline{AC}$	Length of Altitude
3			
3			
3			
3			
4			
4			
4			
4			
5			
5			
5			
5			

Conjectures

Conjectures

Starting at vertex A, number the points of the subdivisions consecutively from 1 to (n – 1). This is an example for n = 5.



Part A

Let a_i be the number that represents the length of the perpendicular from the i'th position of the subdivision along $\overline{AC}$; let b_j be the number that represents the length of the perpendicular from the j'th position of the subdivision along $\overline{AB}$ for $j = n - i$.

- $a_i + b_j = AD$ for any i, $i = 1$ to $(n - 1)$
- For any i, $a_i = b_j$ in any triangle

Part B

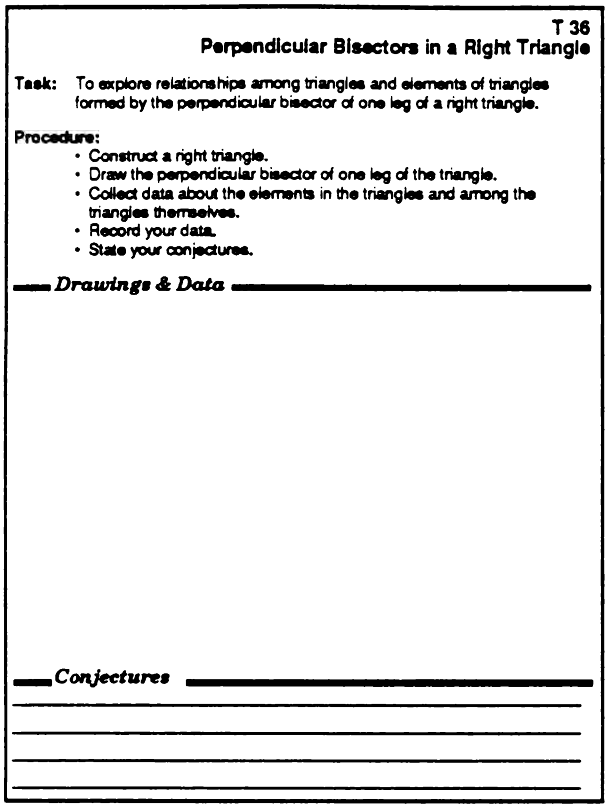
In any triangle, the sum of the lengths of the perpendicular segments on each side of the altitude are equal.

For n = 3, the sum of the lengths of the perpendicular segments on each side of the altitude equals the length of the altitude.

For n > 3, the "sums" are greater than the length of the altitude.

Comments

T 36
Perpendicular Bisectors in a Right Triangle
Teacher Notes



T 36
Perpendicular Bisectors in a Right Triangle

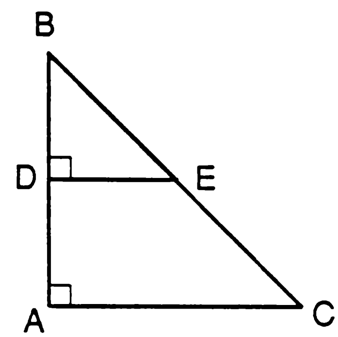
Task: To explore relationships among triangles and elements of triangles formed by the perpendicular bisector of one leg of a right triangle.

Procedure:
- Construct a right triangle.
- Draw the perpendicular bisector of one leg of the triangle.
- Collect data about the elements in the triangles and among the triangles themselves.
- Record your data.
- State your conjectures.

Drawings & Data

Conjectures

Conjectures



- $\angle BED = \angle BCA$
- $\overline{AC} \parallel \overline{DE}$
- Area of $\Delta EDB = \frac{1}{4}$ area of ABC
- Perimeter of EDB $= \frac{1}{2}$ perimeter of ABC
- $DE = \frac{1}{2} AC$
- Area of AED $= \frac{1}{4}$ area of ABC
- $\angle ACE + \angle CAD + \angle ADE + \angle DEC = 360°$
- D is the midpoint of $\overline{AB}$

Comments

Two Perpendicular Bisectors in a Right Triangle

Teacher Notes

T 37

Two Perpendicular Bisectors in a Right Triangle

Task: To explore relationships among triangles and elements of triangles formed by the perpendicular bisectors of both legs of a right triangle.

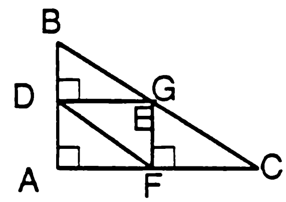
Procedure:

- Construct a right triangle.
- Draw the perpendicular bisector of each leg of the triangle.
- Measure and record data.
- State your conjectures.

Drawings & Data

Conjectures

Conjectures



- $\dfrac{\text{Area of } \triangle ABC}{\text{Area of } ADGF} = 2$
- ADGF is a square if $\triangle ABC$ is an isosceles right triangle.
- $\triangle BDE \sim \triangle GFC$
- $\triangle BDE \cong \triangle GFC$ if and only if $\triangle ABC$ is an isosceles right triangle.
- The perpendicular bisectors meet at the midpoint of the hypotenuse.
- $\dfrac{\text{Area of } \triangle ABC}{\text{Area of } \triangle ADF} = \dfrac{\text{Area of } \triangle ABC}{\text{Area of } \triangle DEF}$

  $= \dfrac{\text{Area of } \triangle ABC}{\text{Area of } \triangle DBG}$

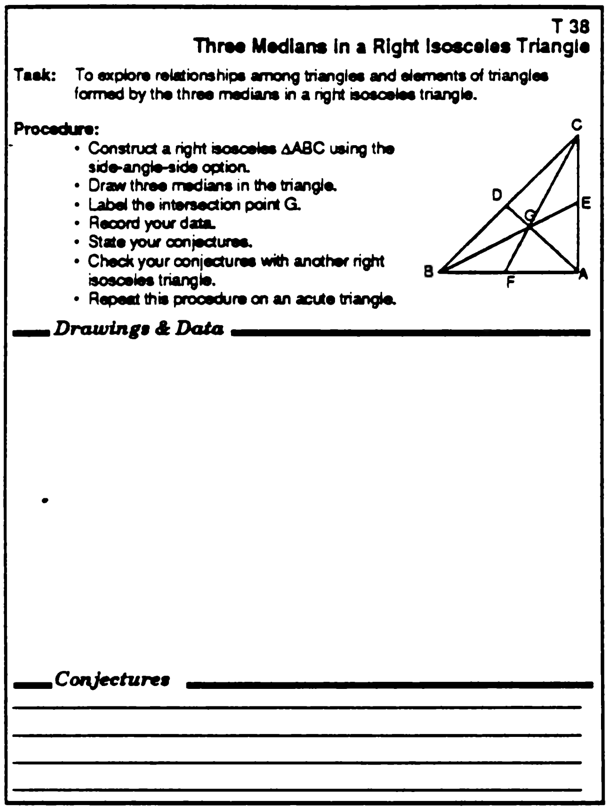
  $= \dfrac{\text{Area of } \triangle ABC}{\text{Area of } \triangle GCF}$

  $= 4$

Comments

Three Medians in a Right Isosceles Triangle

Teacher Notes



T 38

Three Medians in a Right Isosceles Triangle

Task: To explore relationships among triangles and elements of triangles formed by the three medians in a right isosceles triangle.

Procedure:

- Construct a right isosceles $\triangle ABC$ using the side-angle-side option.
- Draw three medians in the triangle.
- Label the intersection point G.
- Record your data.
- State your conjectures.
- Check your conjectures with another right isosceles triangle.
- Repeat this procedure on an acute triangle.

Drawings & Data

Conjectures

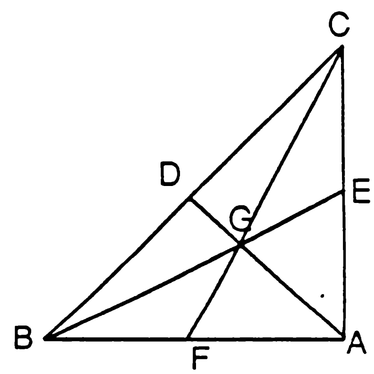
Conjectures

Right Isosceles Triangle:

- $CF = BE$, $AD = DB = CD$
- $BG = 2 \cdot GE$, $AG = 2 \cdot GD$, $GC = 2 \cdot GF$
- The areas of each of the six triangles inside $\triangle ABC$ are equal.
- Area of $\triangle BEA$ = Area of $\triangle CFA$; $\triangle BEA \cong \triangle CFA$
- $\triangle GDC$ and $\triangle GDB$ are congruent right triangles
- $\dfrac{\text{Area of } \triangle BGA}{\text{Area of } \triangle BGD} = \dfrac{\text{Area of } \triangle CGA}{\text{Area of } \triangle CDG} = \dfrac{\text{Area of } \triangle AGC}{\text{Area of } \triangle FGA} = 2$
- Perimeter of $\triangle CDG$ = Perimeter of $\triangle BDG$
- $\dfrac{\text{Perimeter of } \triangle CGE}{\text{Perimeter of } \triangle GEA} = \dfrac{\text{Perimeter of } \triangle BGF}{\text{Perimeter of } \triangle GFA} = 1.2$

Proof

Prove $\dfrac{\text{perimeter of } \triangle CGE}{\text{perimeter of } \triangle GEA} = 1.2$



Suppose $AB = AC = 2x$.

Then $BC = 2\sqrt{2}\,x$ and $EB = \sqrt{5}\,x$, $CF = \sqrt{5}\,x$, $AD = \sqrt{2}\,x$.

Since $\overline{EB}$, $\overline{CF}$, and $\overline{AD}$ are medians,

$CG = BG = \dfrac{2\sqrt{5}\,x}{3}$; $EG = GF = \dfrac{\sqrt{5}\,x}{3}$; $AG = \dfrac{2\sqrt{2}\,x}{3}$ and $GD = \dfrac{\sqrt{2}\,x}{3}$.

$$\frac{\text{Perimeter of CGE}}{\text{Perimeter of GEA}} = \frac{\dfrac{2\sqrt{5}\,x}{3} + \dfrac{\sqrt{5}\,x}{3} + x}{\dfrac{2\sqrt{2}\,x}{3} + \dfrac{\sqrt{5}\,x}{3} + x} = \frac{(\sqrt{5}+1)\,x}{\left[\dfrac{2\sqrt{2}+\sqrt{5}}{3} + 1\right] x} = \frac{(\sqrt{5}+1)\,x}{\left[\dfrac{2\sqrt{2}+\sqrt{5}+3}{3}\right] x}$$

$$= \frac{3 + 3\sqrt{5}}{3 + 2\sqrt{2} + \sqrt{5}}$$

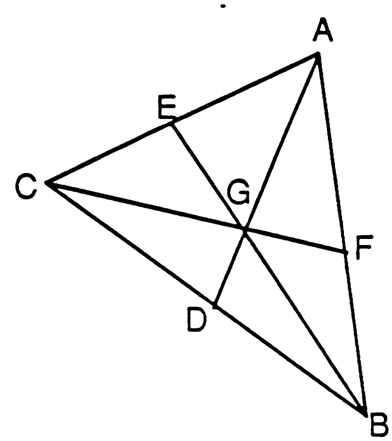
Comments

Three Medians in a Right Isosceles Triangle

Teacher Notes

Conjectures

<u>Acute Triangle:</u>



- The areas of each of the six triangles in ΔABC are equal.
- $CG = 2 \cdot GF$, $BG = 2 \cdot GE$, $AG = 2 \cdot GD$
- $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = 4$
- $\dfrac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta DEF} = 2$
- Many true facts about midsegments $\overline{DF}$, $\overline{FE}$, and $\overline{DE}$.
- In equilateral ΔABC, $\angle AGB = \angle BGC = \angle CGA = 120°$.

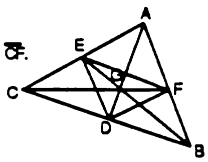
Comments

T 39

Angle Bisectors in Isosceles Triangles

Task: To explore figures formed by the three angle bisectors in isosceles triangles.

Procedure:
- Construct an isosceles ΔABC.
- Draw the three angle bisectors $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.
- Draw $\overline{DE}$, $\overline{EF}$, and $\overline{DF}$.
- Label the intersection point of the angle bisectors with point G.
- Measure and record data.
- State your conjectures.



Drawings & Data

Conjectures

T 39
Angle Bisectors in Isosceles Triangles
Teacher Notes

Conjectures

- $\Delta CFB \cong \Delta BEC$ by SAS; their areas and perimeters are equal.
- $BF = CE$, $BE = CF$, and $\angle CEB = \angle BFC$ by corresponding parts of congruent triangles.
- $AE = AF$ and $\angle AEF = \angle AFE$.
- ΔCGB is isosceles; $CG = BG$.
- $\angle AEF = \angle AFE$, $\angle CEB = \angle CFB$ implies that $\angle GEF = \angle GFE$. Thus $EG = FG$ and ΔEFG is isosceles.
- $EG = FG$, $\angle EGD = \angle FGD$ (vertical angles), and $GD = GD$ implies that $\Delta EGD \cong \Delta FGD$ by SAS. Thus, $ED = FD$ and ΔEFD is an isosceles triangle.

Notes

This problem contains many conjectures related to isosceles, right, and congruent triangles. Labeling additional points of intersection in the diagram can lead to the exploration of other relationships. For example, label the intersection of $\overline{AD}$ and $\overline{EF}$ with H. $\angle AHE = \angle AHF = 90°$ and $\overline{EF} \parallel \overline{BC}$ because $\overline{AD} \perp \overline{BC}$.

Comments

T 40
Three Altitudes in Acute Triangles
Parts A and B
Teacher Notes

T 40
Three Altitudes in Acute Triangles - Part A

Task: To explore relationships in figures formed by the three altitudes in acute triangles.

Procedure:
- Construct an acute $\triangle ABC$.
- Draw the three altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.
- Draw $\overline{EF}$.
- Label the intersection point of the altitudes with point G.
- Measure and record your data about the relationships between the various angles and triangles in the drawing.
- State your conjectures.

Drawings & Data

Conjectures

T 40
Three Altitudes in Acute Triangles - Part B

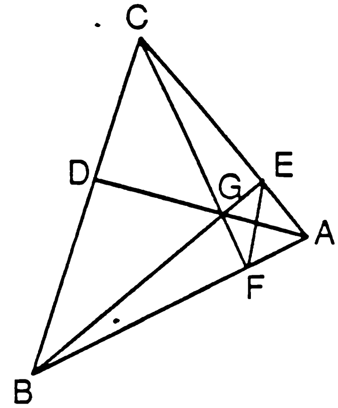
Task: To explore relationships in figures formed by the three altitudes in acute triangles.

Procedure:
- Use your construction from Part A:
  - Construct an acute $\triangle ABC$.
  - Draw the three altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.
  - Draw $\overline{EF}$.
  - Label the intersection point of the altitudes with point G.
- Draw $\overline{DE}$ and $\overline{DF}$.
- State your conjectures about the relationships between various angles and triangles in the drawing.

Drawings & Data

Conjectures

Conjectures



Part A

- $\Delta FCA \sim \Delta GCE$
  because $\angle CAF = \angle CGE$ and $\angle FCA = \angle GCE$.

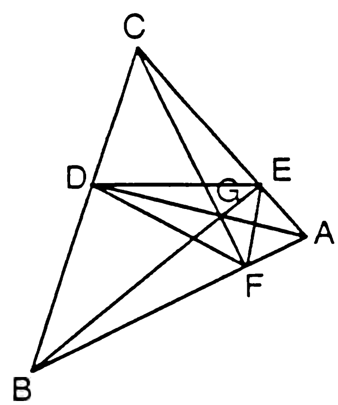
- $\Delta FGA \sim \Delta CGD$
  because $\angle FGA = \angle CGD$ and $\angle GAF = \angle DCG$.

- $\Delta ADC \sim \Delta BEC$
  because $\angle CAD = \angle CBE$ and $\angle DCA = \angle ECB$.

- $\Delta BGF \sim \Delta CGE$
  $\Delta AGE \sim \Delta BCE$
  $\Delta DGB \sim \Delta EGA$

Part B

- Similar relationships hold when $\overline{DE}$ and $\overline{DF}$ are drawn.



Comments

Three Altitudes in Acute Triangles Part C

Teacher Notes

T 40

Three Altitudes in Acute Triangles - Part C

Task: To explore relationships in figures formed by the three altitudes in acute triangles.

Procedure:

- Use your construction from Part B:
  - Construct an acute ΔABC.
  - Draw the three altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.
  - Draw $\overline{EF}$.
  - Label the intersection point of the altitudes with point G.
  - Draw $\overline{DE}$ and $\overline{DF}$.
- Circumscribe ΔBFE.
- State your conjectures.

Drawings & Data

Conjectures

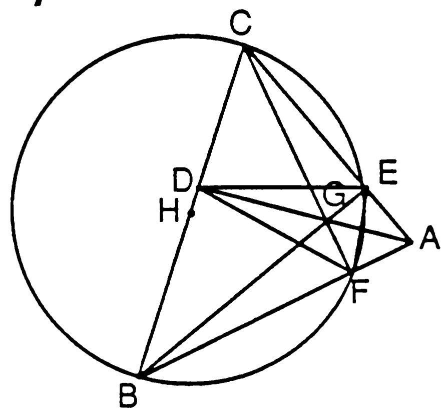
How does this circle relate to ΔABC?

Conjectures

Part C

- The circle that circumscribed ΔBFE also passes through vertex C.
- The center H of this circle lies on $\overline{CB}$.
- If H is the center, then CH = BH = HF = HE

Proof



$\angle BEC = 90°$ (BE is an altitude).
$\overline{CB}$ is a diameter of a circle that circumscribes ΔCEB.

$\angle CFB = 90°$ (the same).
$\overline{CB}$ is a diameter of a circle that circumscribes ΔBFE.
$\overline{CB}$ is a common diameter so the two circles are the same circle.

Comments

Three of Everything

Teacher Notes

T 41

Three of Everything

Task: To explore the relationships among the points of intersection of the three **medians**, **altitudes**, **angle bisectors**, and **perpendicular bisectors** in triangles.

Procedure:

- Construct a right triangle.
- Draw the three medians in the triangle.
- Place a check mark under the columns which describe the intersection of the three **medians**.
- Repeat the procedure for other triangles and check the appropriate column on the chart.
- Use the same procedure to complete the chart below for **altitudes**, **angle bisectors**, and **perpendicular bisectors**.
- State your conjectures.

The three **medians** intersect...

Triangle Type	In One Point	Outside Triangle	Inside Triangle	On Triangle
Right				
Acute				
Obtuse				
Isosceles (Acute)				
Isosceles (Obtuse)				
Equilateral				

The three **altitudes** intersect...

Triangle Type	In One Point	Outside Triangle	Inside Triangle	On Triangle
Right				
Acute				
Obtuse				
Isosceles (Acute)				
Isosceles (Obtuse)				
Equilateral				

Data

The three **medians** intersect...

Triangle Type	In One Point	Outside Triangle	Inside Triangle	On Triangle
Right	x		x	
Acute	x		x	
Obtuse	x		x	
Isosceles (Acute)	x		x	
Isosceles (Obtuse)	x		x	
Equilateral	x		x	

The three **altitudes** intersect...

Triangle Type	In One Point	Outside Triangle	Inside Triangle	On Triangle
Right	x			x
Acute	x		x	
Obtuse	x	x		
Isosceles (Acute)	x		x	
Isosceles (Obtuse)	x	x		
Equilateral	x		x	

Comments

T 41 (page 2)
Three of Everything

The three **angle bisectors** intersect...

Triangle Type	In One Point	Outside Triangle	Inside Triangle	On Triangle
Right				
Acute				
Obtuse				
Isosceles (Acute)				
Isosceles (Obtuse)				
Equilateral				

The three **perpendicular bisectors** intersect...

Triangle Type	In One Point	Outside Triangle	Inside Triangle	On Triangle
Right				
Acute				
Obtuse				
Isosceles (Acute)				
Isosceles (Obtuse)				
Equilateral				

Conjectures

Data

The three **angle bisectors** intersect...

Triangle Type	In One Point	Outside Triangle	Inside Triangle	On Triangle
Right	x		x	
Acute	x		x	
Obtuse	x		x	
Isosceles (Acute)	x		x	
Isosceles (Obtuse)	x		x	
Equilateral	x		x	

The three **perpendicular bisectors** intersect...

Triangle Type	In One Point	Outside Triangle	Inside Triangle	On Triangle
Right	x			x
Acute	x		x	
Obtuse	x	x		
Isosceles (Acute)	x		x	
Isosceles (Obtuse)	x	x		
Equilateral	x		x	

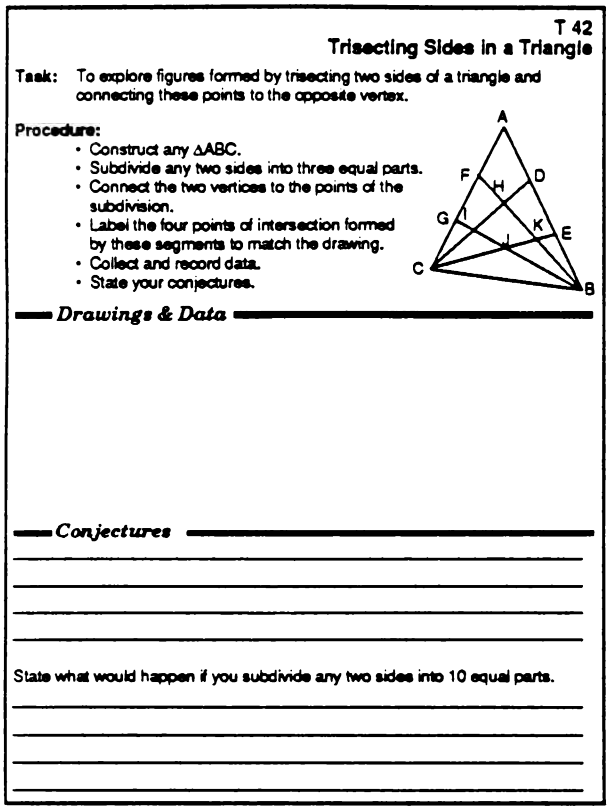
Notes

This could serve either as a summary project on concurrency in triangles or as the basis of further discussions on properties and theorems regarding intersections of elements.

Comments

Trisecting Sides in a Triangle

Teacher Notes



T 42

Trisecting Sides in a Triangle

Task: To explore figures formed by trisecting two sides of a triangle and connecting these points to the opposite vertex.

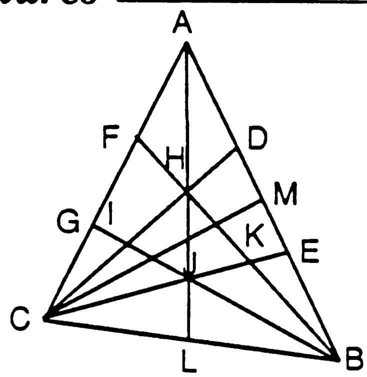
Procedure:
- Construct any ΔABC.
- Subdivide any two sides into three equal parts.
- Connect the two vertices to the points of the subdivision.
- Label the four points of intersection formed by these segments to match the drawing.
- Collect and record data.
- State your conjectures.

Drawings & Data

Conjectures

State what would happen if you subdivide any two sides into 10 equal parts.

Conjectures



- The median from A to $\overline{BC}$ passes through points H and J.

- Draw median $\overline{CM}$ to $\overline{AB}$.
  Area of ΔCEM = Area of ΔCMD

- Area of HIGF = Area of EKHD

- Area of ΔBHA = Area of ΔCHA
  Area of ΔBKE = Area of ΔCIG
  Area of ΔACJ = Area of ΔBJA

- $\dfrac{AJ}{JL} = 4$

If sides are divided into 10 equal parts:

- The median from A to $\overline{BC}$ passes through the points of intersection as above.

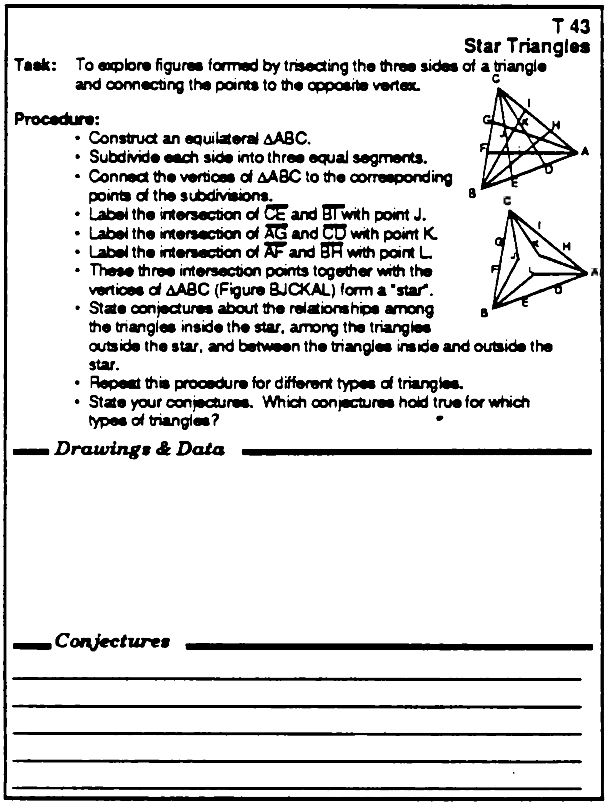
- Similar results involving area hold as above.

- Suppose the $\overline{AB}$ and $\overline{AC}$ are divided into n = 3, 4, 5, 6. . . equal parts and $\overline{AJ_n}$, n = 3, 4, . . . represents that <u>portion of the median</u> from A to $\overline{BC}$ where J_n is the intersection of the median with the segment drawn from C to $\overline{AB}$ closest to B. For example, when n = 6, note $\overline{AJ_n}$ below.

Comments

Star Triangles

Teacher Notes



T 43
Star Triangles

Task: To explore figures formed by trisecting the three sides of a triangle and connecting the points to the opposite vertex.

Procedure:

- Construct an equilateral ΔABC.
- Subdivide each side into three equal segments.
- Connect the vertices of ΔABC to the corresponding points of the subdivisions.
- Label the intersection of $\overline{CE}$ and $\overline{BI}$ with point J.
- Label the intersection of $\overline{AG}$ and $\overline{CD}$ with point K.
- Label the intersection of $\overline{AF}$ and $\overline{BH}$ with point L.
- These three intersection points together with the vertices of ΔABC (Figure BJCKAL) form a "star".
- State conjectures about the relationships among the triangles inside the star, among the triangles outside the star, and between the triangles inside and outside the star.
- Repeat this procedure for different types of triangles.
- State your conjectures. Which conjectures hold true for which types of triangles?

Drawings & Data

Conjectures

Conjectures

- Area of ΔBJA = Area of ΔBKC = Area of ΔAJC
- $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta BLA} = 5; \quad \dfrac{\text{Area of } \Delta BLA}{\text{Area of } \Delta KLJ} = 5$

Note: This is another method for dividing a triangle into five sections of equal area.

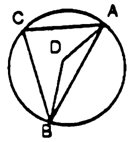
- $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta KLJ} = 25$
- $\dfrac{AB}{KL} = \dfrac{BC}{LJ} = \dfrac{AC}{KJ} = 5$
- $\dfrac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta KLJ} = 5$
- ΔKLJ is equilateral. If ABC is an "x" triangle, then KLJ is an "x" triangle and the two triangles are similar.
- $\dfrac{\text{Area of "star"}}{\text{Area of } \Delta ABC} = .4$
- All of the conjectures listed for the above hold true for all types of triangles.

Comments

T 44
Circumscribed Triangles
Parts A and B
Teacher Notes

T 44
Circumscribed Triangles - Part A

Task: To explore the relationship between a central angle and angles on an arc.



Procedure:
- Construct an acute $\triangle ABC$.
- Draw a circumscribed circle with center D.
- Draw $\overline{AD}$ and $\overline{BD}$.
- Record your data.
- State as many conjectures as you can about the relationship of $\angle BDA$ to other angles in the drawing.

Drawings & Data

Conjectures

T 44
Circumscribed Triangles - Part B

Task: To explore the relationship between a central angle and angles on an arc.

Procedure:
- Use your construction from Part A.
  - Construct an acute $\triangle ABC$.
  - Draw a circumscribed circle with center D.
  - Draw $\overline{AD}$ and $\overline{BD}$.
- Use the Adjustable element option on the Measure menu to locate points E and F on arc ABC. Locate E and F so that DE = DF = AD.
- Record your data.
- State your conjectures.
- Which conjectures are identical to those found in Part A?

Drawings & Data

Conjectures

Conjectures

Part A

- $\angle BDA = 2 \cdot \angle BCA$

Proof

$CD = AD = BD$ (Circumscribed circle with center D)
In $\triangle ABC$, $2 \cdot \angle ACD + 2 \cdot \angle DCB + 2 \cdot \angle DBA = 180°$,
so $2 \cdot \angle ACD + 2 \cdot \angle DCB = 180° - 2 \cdot \angle DBA$
In $\triangle BDA$, $\angle BDA + 2 \cdot \angle DBA = 180°$
so $\angle BDA = 180° - 2 \cdot \angle DBA$
Thus, $2 \cdot \angle ACD + 2 \cdot \angle DCB = \angle BDA$,
or $2(\angle ACD + \angle DCB) = \angle BDA$
$2 \cdot \angle BCA = \angle BDA$

Conjectures

Part B

- The same conjecture holds for any point on $\widehat{ACB}$.

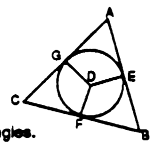
Comments

T 45
A Triangle and Its Inscribed Circle
Teacher Notes

T 45
A Triangle and Its Inscribed Circle

Task: To explore the relationship between the center of an inscribed circle and the three tangent points to the triangle in which it is inscribed.

Procedure:
- Construct an equilateral $\triangle ABC$.
- Inscribe a circle with center D.
- Draw the radius to each of the tangent points.
- Describe the relationships between the intersection points (E, F, G) and the elements and properties of the triangle.
- Record your data.
- Repeat the construction on other types of triangles.
- State your conjectures on the following page.



Drawings & Data

T 45 (page 2)
A Triangle and Its Inscribed Circle

Conjectures

Describe the relationship between the center of the circle D and the sides of the triangle; between the center D and the vertices; and between the center D and other properties of the triangle.

Explore the properties of the triangles formed by the points E, F, and G and the vertices A, B, and C.

Which conjectures for equilateral triangles are true for other types of triangles as well?

Conjectures

<u>Relationships with the center D in equilateral triangle:</u>
- D is equidistant from E, F, and G
- D is also the point where altitudes, angle bisectors, and perpendicular bisectors intersect

<u>Properties of E, F, and G:</u>
- The lengths from each vertex to the tangent points are equal
- The distances from the tangent points to point D are equal
- The length of the radius of the circle is 1/3 the length of an altitude of $\triangle ABC$
- The tangent points are the midpoints of each side

<u>Triangles formed by vertices A, B, and C and points E, F, and G:</u>
- The six triangles ($\triangle AEF$, $\triangle BFG$, $\triangle CGE$, $\triangle DEF$, $\triangle DGF$, $\triangle DGE$) are isosceles
- There are three pairs of congruent right triangles

<u>Conjectures that hold for all types of triangles:</u>
- The distances from D to E, F, and G are equal
- AE = AG, BE = BF, CG = CF

Comments

The Center of an Inscribed Circle Parts A and B

Teacher Notes

T 46

The Center of an Inscribed Circle - Part A

Task: To explore a figure formed by an inscribed circle in a triangle.

Procedure:
- Construct a $\triangle ABC$ using the side-side-side option with each side 5 units long.
- Inscribe a circle in $\triangle ABC$.
- Record data.
- State your conjectures about the relationship of the center D to $\triangle ABC$.
- Repeat this procedure for other types of triangles.
- For each type of triangle, state your conjectures.

Drawings & Data

Conjectures

T 46

The Center of an Inscribed Circle - Part B

Task: To construct the center of the inscribed circle of a triangle.

Procedure:
- Construct an equilateral $\triangle ABC$.
- Find the center of the inscribed circle **without** using the "Inscribed Circle" option.
- Describe your method.
- Try this method on different types of triangles.
- State your conjectures.

Drawings & Data

Conjectures

Conjectures

Part A

Equilateral Triangles:

- DA = DB = DC
- $\angle ACD = \angle BCD = \angle CBD = \angle DBA = \angle BAD = \angle DAC$
- $\angle CDB = \angle BDA = \angle ADC$
- D is equidistant to each side
- D is the intersection point of the medians
- D is the intersection point of the perpendicular bisectors

All Triangles:

- $\angle CAD = \angle BAD$; $\angle ABD = \angle CBD$; $\angle BCD = \angle DCA$
- D is equidistant to each side

Part B

The center is the intersection of the angle bisectors. The radius is the distance from the point of intersection to any side of the triangle.

Comments

The Center of a Circumscribed Circle
Parts A and B

Teacher Notes

T 47

The Center of a Circumscribed Circle - Part A

Task: To explore the relationship between the center of a circumscribed circle and the triangle which it circumscribes.

Procedure:

- Construct a ΔABC using the side-side-side option with each side 5 units long.
- Circumscribe a circle around ΔABC.
- State your conjectures about the relationship of the center D to ΔABC.
- Repeat this procedure for other types of triangles.
- For each type of triangle, state your conjectures.

Drawings & Data

Conjectures

T 47

The Center of a Circumscribed Circle - Part B

Task: To construct the center of the circumscribed circle of a triangle.

Procedure:

- Construct an equilateral ΔABC.
- Find the center of the circumscribed circle **without** using the "Circumscribed Circle" option.
- Describe your method.
- Try this method on different types of triangles.
- State your conjectures.

Drawings & Data

Conjectures

Conjectures

Part A

Equilateral Triangle:

- DA = DB = DC
- $\angle ACD = \angle BCD = \angle CBD = \angle DBA = \angle BAD = \angle DAC$
- $\angle CDB = \angle BDA = \angle ADC$
- D is equidistant to each side
- D is the intersection point of the angle bisectors
- D is the intersection point of the medians
- The radius is two-thirds of any median

Other Triangles:

- DA = DB = DC
- $\angle DBA = \angle DAB$; $\angle DAC = \angle DCA$; $\angle DCB = \angle DBC$
- ΔDBA, ΔDCA, and ΔDCB are isosceles
- If ΔBAC is a right triangle, then D is the midpoint of hypotenuse BC.
- If ΔABC is obtuse, then D is in the exterior of ΔABC.
- If ΔABC is acute, then D is in the interior of ΔABC.

Part B

- The center is the intersection of the perpendicular bisectors. The radius is the distance from the point of intersection to any vertex of the triangle.
- This method works on all triangles.

Comments

T 48
Three Consecutive Integers
Teacher Notes

T 48
Three Consecutive Integers

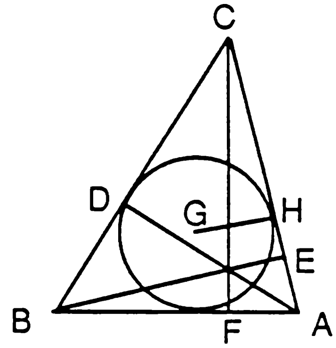
Task: To explore the relationship between altitudes and the radius of the inscribed circle in triangles whose side lengths are three consecutive integers.

Procedure:
- Construct a triangle using the side-side-side option with lengths that are three consecutive integers.
- Draw the three altitudes.
- Now draw an inscribed circle in the triangle.
- Investigate the relationships between the altitudes of the triangle and the radius of the circle.
- Record your data.
- State your conjectures.

Drawings & Data

Conjectures

Conjectures



- $\overline{GH} \parallel \overline{BE}$
  Lines that are perpendicular to the same line are parallel.

- $BE = 3 \cdot GH$
  This relationship holds for only one altitude. The altitude to the side whose length is represented by the second consecutive integer is three times the radius. In general, there is a radius which is parallel to each altitude.

Proof

Let a = length of $\overline{AB}$ (shortest side), r = GH, and h = BE.

Area of $\Delta ABC = \frac{(a+1)\cdot h}{2}$

Area of ΔABC = Area of ΔBGA + Area of ΔAGC + Area of ΔBGC

$$= \frac{a\cdot r}{2} + \frac{(a+1)\cdot r}{2} + \frac{(a+2)\cdot r}{2}$$

$$= \frac{r}{2}\cdot(3a+3) = \frac{3r\cdot(a+1)}{2}$$

So, $\frac{\cancel{(a+1)}\cdot h}{2} = \frac{3\cdot r\cancel{(a+1)}}{2}$ and hence, $h = 3r$

Comments

Reflecting Altitudes

Teacher Notes

T 49
Reflecting Altitudes

Task: To explore the results of reflecting the altitudes of triangles.

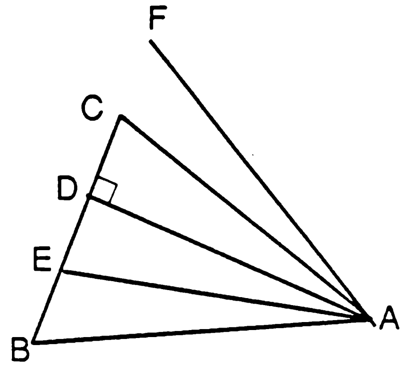
Procedure:

- Construct an acute $\triangle ABC$.
- Draw an altitude from vertex A.
- Reflect $\overline{AC}$ in $\overline{AD}$ to form $\overline{AE}$.
- Reflect $\overline{AB}$ in $\overline{AD}$ to form $\overline{AF}$.
- Repeat this construction on an obtuse triangle.
- Predict under what conditions $\overline{AE}$ will overlap $\overline{AB}$?
- Under what conditions will $\overline{AF}$ overlap $\overline{AC}$?
- State your conjectures.

Triangle Type	Does $\overline{AE}$ overlap $\overline{AB}$?	Does $\overline{AF}$ overlap $\overline{AC}$?
Acute		
Right		
Obtuse		
Isosceles (Acute)		
Isosceles (Obtuse)		
Equilateral		

Conjectures

Conjectures



- $\overline{AE}$ and $\overline{AF}$ *will not* necessarily coincide with $\overline{AB}$ and $\overline{AC}$ respectively.

- For isosceles triangles with vertex angle A and for equilateral triangles, $\overline{AE}$ and $\overline{AF}$ *will* coincide with $\overline{AB}$ and $\overline{AC}$ respectively.

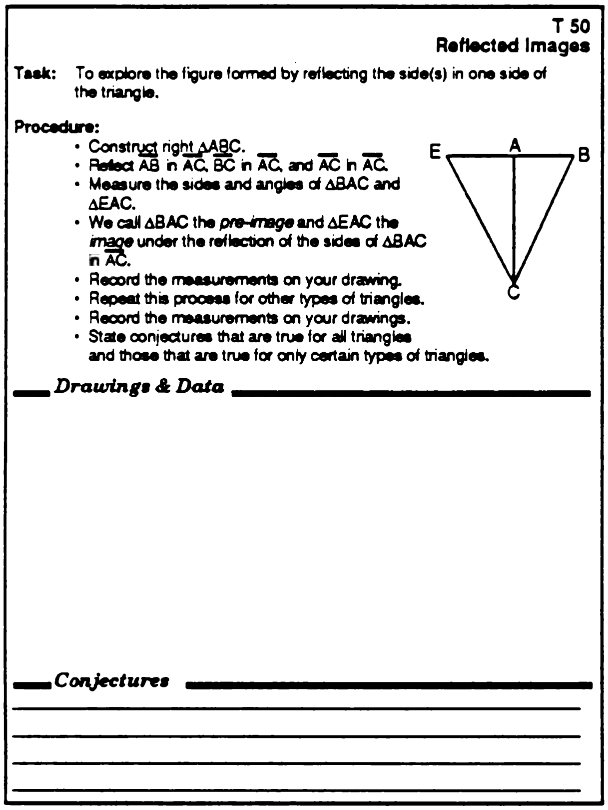
Notes

The purpose of this exercise is for students to examine under what conditions an altitude in a triangle acts as a line of symmetry.

Comments

Reflected Images

Teacher Notes



T 50
Reflected Images

Task: To explore the figure formed by reflecting the side(s) in one side of the triangle.

Procedure:

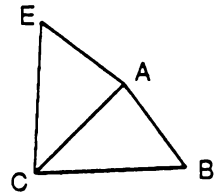
- Construct right ΔABC.
- Reflect $\overline{AB}$ in $\overline{AC}$, $\overline{BC}$ in $\overline{AC}$, and $\overline{AC}$ in $\overline{AC}$.
- Measure the sides and angles of ΔBAC and ΔEAC.
- We call ΔBAC the *pre-image* and ΔEAC the *image* under the reflection of the sides of ΔBAC in $\overline{AC}$.
- Record the measurements on your drawing.
- Repeat this process for other types of triangles.
- Record the measurements on your drawings.
- State conjectures that are true for all triangles and those that are true for only certain types of triangles.

Drawings & Data

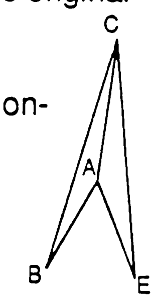
Conjectures

Conjectures

- $CB = CE$, $AB = AE$
- $\angle BCA = \angle ECA$, $\angle BAC = \angle EAC$, $\angle ABC = \angle AEC$
- $\Delta ABC \cong \Delta AEC$
- Area of ΔABC = Area of ΔAEC
- Perimeter of ΔABC = Perimeter of ΔAEC
- If ΔBAC is a right triangle, then points A, B, and E are on the same line. This is only true for right triangles.



- When ΔBAC is not a right triangle, then the original shape and the reflection create a kite.
- When ΔABC is obtuse, the kite AECB is non-convex.
- $\overline{BE}$ is always perpendicular to $\overline{AC}$.



- The closer $\angle BAC$ is to 90°, the nearer A is to EB.

Notes

This exercise should demonstrate that the measurements of the sides and angles in ΔBAC (pre-image) are equal to the measurements of the corresponding sides and angles in ΔEAC (image).

That is,

vertex B corresponds to vertex E
vertex A corresponds to vertex A
vertex C corresponds to vertex C under the given reflection.

This is an easy problem that could be done with mirrors as well as with the SUPPOSER. Data for answering this problem can be qualitative and/or quantitative. Some conjectures (such as the last in the list) can be derived by analyzing the visual changes in different types of triangles and looking at the types of shapes that are created by reflection. This could be done even before learning about quadrilaterals. Other conjectures can be derived by looking at measurements and can be used to learn about or review the properties of quadrilaterals. Naturally, all reflections yield symmetric shapes and this could also be a topic for discussion.

Comments

T 51
Reflection - Part A

Task: To explore reflection in triangles.

Procedure:
- Construct an acute ΔABC.
- Reflect ΔABC in $\overline{AB}$ by reflecting $\overline{BC}$ in $\overline{AB}$, $\overline{AC}$ in $\overline{AB}$, and $\overline{AB}$ in $\overline{AB}$.
- Record the drawing.
- State your conjectures.

Drawings & Data

Conjectures

Are the two triangles congruent?

T 51
Reflection - Part B

Task: To explore reflection in triangles.

Procedure:
- Construct any ΔABC.
- Use reflections to draw another triangle that shares only one point with ΔABC.
- What point do the two triangles share?
- Record the drawing.
- State your conjectures.

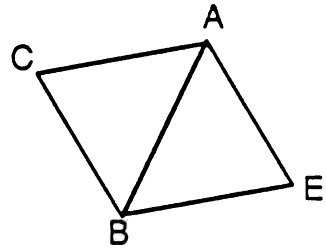
Drawings & Data

Conjectures

Are the two triangles congruent?

Conjectures

Part A

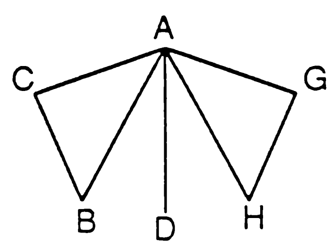


ΔABC and ΔABE share $\overline{AB}$ in common.

Part B

Here are different ways to create a shape that shares only a point with the original triangle:

- Draw reflecting line $\overline{DE}$ through A, parallel to $\overline{BC}$.
- Extend $\overline{AB}$ from A. Bisect $\angle DAC$. $\overline{AE}$ is a reflecting line.
- Draw median $\overline{AD}$ in ΔABC from vertex A. Reflect $\overline{AB}$ and $\overline{AC}$ in $\overline{AD}$. ΔAEF has point A in common with ΔABC; however, the two triangles are overlapping.
- Any line which passes through any vertex of ΔABC, and does not have any other points in common with ΔABC, could be a reflecting line for two triangles that share a point in common.



Comments

T 51
Reflection - Part C

Task: To explore reflection in triangles.

Procedure:
- Construct any ΔABC.
- Use reflections to draw another triangle that shares no points with ΔABC.
- Record the drawing.
- State your conjectures.

Drawings & Data

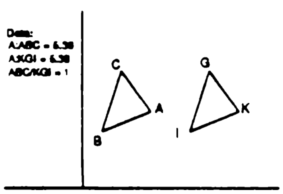
Conjectures

Are the two triangles congruent?

T 51
Reflection - Part D

Task: To explore reflection in triangles.

Procedure:
- Construct any ΔABC.
- Develop a method for "sliding" ΔABC to a new position such that ΔABC and its image are congruent and look like the diagram below:



- Record the drawing.
- State your conjectures.

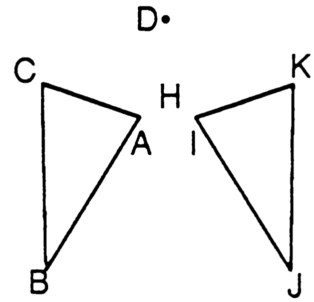
Drawings & Data

Conjectures

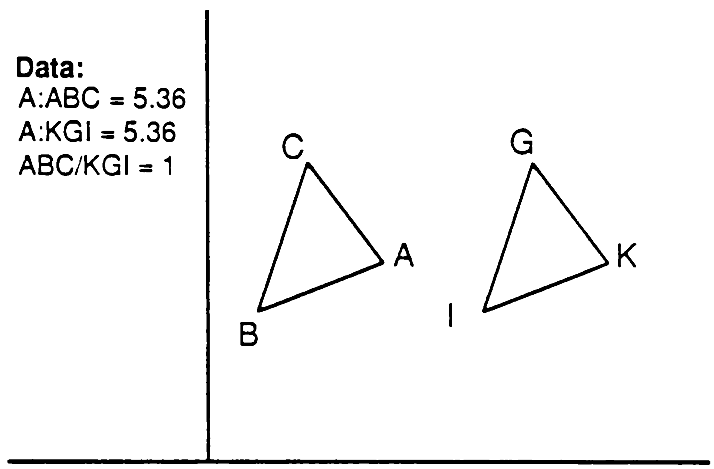
Conjectures

Part C

Reflecting a triangle that shares no points. Any line which does <u>not</u> intersect the triangle will produce a separate image.



Part D



Draw acute ΔABC and $\overline{ED}$ through A, parallel to $\overline{BC}$. Reflect ΔABC in $\overline{ED}$ to obtain ΔHGI by reflecting $\overline{AC}$ in $\overline{ED}$ and $\overline{AB}$ in $\overline{ED}$. Reflect ΔHGI in $\overline{GI}$ to obtain ΔKGI. Erase all segments except those that make up ΔABC and ΔKGI. $\Delta ABC \cong \Delta KGI$; check by measuring the sides and angles of ΔABC and ΔKGI.

Notes

$\overline{ED} \parallel \overline{IG}$ and the "distance" that each point moves is twice the distance between parallel lines.

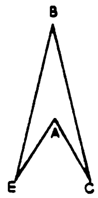
Comments

T 52
Construction Challenges
Teacher Notes

T 52
Construction Challenges

Task: To develop a procedure for reproducing this figure.

Procedure:
- Make a drawing similar to this figure.
- Collect data.
- Describe below the procedures for reproducing this figure.
- State your conjectures.



Drawings & Data

Conjectures

Procedure for reproducing figure:

____

____

____

For quadrilateral ACBE above, under what conditions will $\angle CAE$ be a 90° angle?

____

____

____

Conjectures

- Reflect obtuse ΔBAC in $\overline{AB}$. (The orientation of your shape may be different.)
- Draw obtuse ΔABC using side-angle-side with $\angle BAC = 135°$; then $\angle CAE = 90°$.

Comments

T 53
N-Gons
Teacher Notes

T 53
N-Gons

Task: To use triangles and reflections to construct polygons.

Procedure:
- Using triangles and reflections, develop procedures for constructing the following shapes: square, rectangle, rhombus, kite, pentagon, octagon, nonagon, and decagon.
- On the following page, state your conjectures about constructing these polygons.

Drawings & Data

T 53 (page 2)
N-Gons

Conjectures

Square ______________________

Rectangle ______________________

Rhombus ______________________

Kite ______________________

Pentagon ______________________

Octagon ______________________

Nonagon ______________________

Decagon ______________________

Notes

This exercise can be used to study the properties of n-gons and the congruent triangles that comprise n-gons.

A square is generally presented as the special case of a parallelogram. It is also a regular 4-gon and can be constructed as such.

The more difficult case in the project is the rectangle. A diagonal in a rectangle divides it into two congruent triangles. However, a diagonal is not a symmetry line. Reflections on diagonals produce kites and not rectangles.

Conjectures

Rectangle: Start with any isosceles triangle. Reflect isosceles ΔABC ($AB = AC$) in a line parallel to $\overline{BC}$ through vertex A. BGFC is a rectangle.

Comments

T 54
Reflecting a Point to Create Triangles
Parts A and B
Teacher Notes

T 54
Reflecting a Point to Create Triangles - Part A

Task: To explore the figure formed by reflecting the intersection point of the **altitudes** in each side of a triangle and connecting the three image points.

Procedure:
- Construct an acute $\triangle ABC$.
- Draw the three altitudes.
- Label G as their point of intersection.
- Reflect point G in each of the three sides of $\triangle ABC$ producing points H, I, J.
- Draw $\triangle DEF$ and $\triangle HIJ$.
- State your conjectures about the relationships among the points, elements, and triangles.
- Repeat the procedure for other types of triangles.

Drawings & Data

Conjectures

T 54
Reflecting a Point to Create Triangles - Part B

Task: To explore the figure formed by reflecting the intersection point of the **perpendicular bisectors** of each side of a triangle and connecting the three image points.

Procedure:
- Construct an acute $\triangle ABC$.
- Draw the three perpendicular bisectors.
- Label G as their point of intersection.
- Reflect point G in each of the three sides of $\triangle ABC$ producing points H, I, J.
- Draw $\triangle DEF$ and $\triangle HIJ$.
- State your conjectures about the relationships among the points, elements, and triangles.
- Repeat the procedure for other types of triangles.

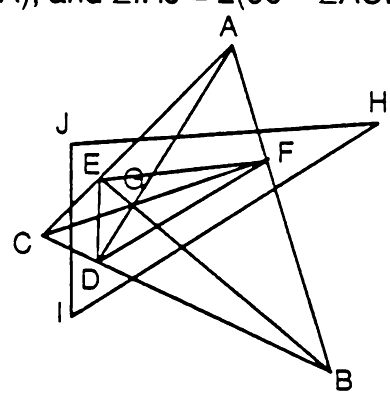
Drawings & Data

Conjectures

Conjectures

Part A

- A circle which circumscribes $\triangle ABC$ passes through points H, I, J.
- $\triangle DFE \sim \triangle IHJ$
- $\dfrac{\text{Perimeter of } \triangle IHJ}{\text{Perimeter of } \triangle DFE} = 2$
- $\dfrac{\text{Area of } \triangle IHJ}{\text{Area of } \triangle DFE} = 4$
- If $\triangle ABC$ is equilateral then $\triangle JHI \cong \triangle ABC$
- If $\triangle ABC$ is acute, then $\angle JIH = 2(90 - \angle CAB)$, $\angle IJH = 2(90 - \angle CBA)$, and $\angle IHJ = 2(90 - \angle ACB)$



Part B

- The triangle constructed by connecting the reflected points (the intersection of the perpendicular bisectors reflected in each side) is congruent to $\triangle ABC$.
- The "reflected" triangle is similar to the triangle formed by the points which are the intersections of the perpendicular bisectors with the sides of $\triangle ABC$; the ratio of their areas is 4:1 and the ratio of their perimeters is 2:1.

Notes

There are many other interesting conjectures related to the positions of the triangles with respect to each other.

Comments

Triangular Sections

Teacher Notes

T 55

Triangular Sections

Line segments divide any triangle into triangular sections. For example, an angle bisector divides a triangle into two triangular sections and three medians divide any triangle into six triangular sections.

Task: To describe methods of subdividing triangles into subtriangles that have equal area.

Procedure:
- Construct any $\triangle ABC$.
- Draw line segments.
- State your conjectures.

Drawings & Data

Conjectures

What kind of segment(s) or combinations of segments and how many of them divide(s) any triangle into **two** sections with equal areas? __________

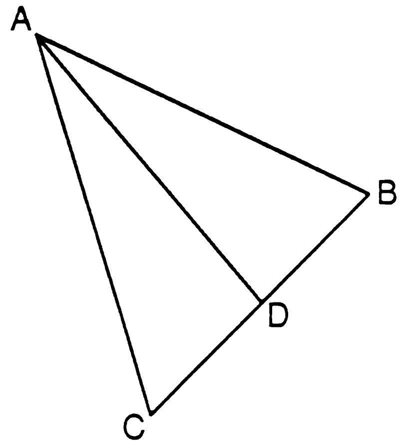
Three sections with equal areas? __________

Four sections with equal areas? __________

Five sections with equal areas? __________

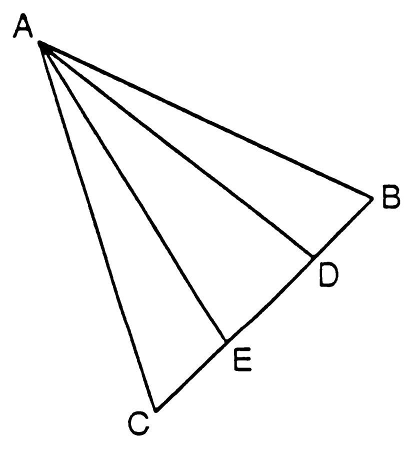
Conjectures

- Two sections with equal areas: any median.

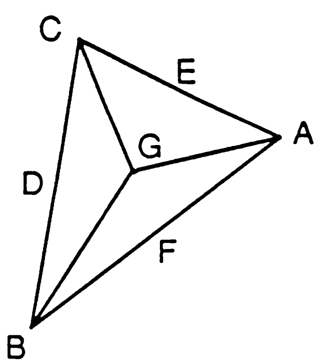


- Three sections with equal areas:

(1) Subdivide any side of a triangle into three sections of the same length, connect the points that label the subdivision to the vertex of the opposite angle. (Subdividing a side into n sections and connecting the points to the vertex of the opposite angle will yield n triangles with equal areas.)



(2) Draw the three medians, label the point where they intersect. Erase $\overline{GD}$, $\overline{GE}$, and $\overline{GF}$.



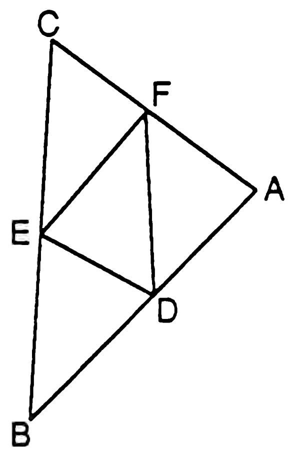
Notes

This problem could also be posed in terms of perimeter.

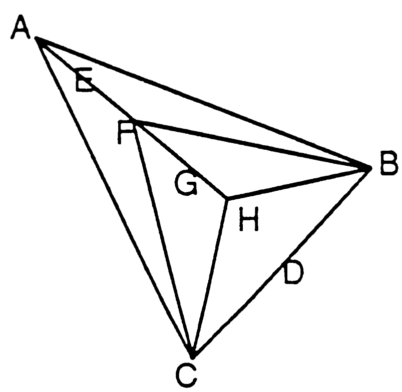
Comments

Conjectures

- Four sections with equal areas:
  (1) Subdivide any side into four equal lengths (see above).
  (2) Draw three midsegments in any triangle.



- Five sections with equal areas:
  (1) Subdivide any side into five equal lengths (as above).
  (2) Draw a median from vertex A; subdivide the median $\overline{AD}$ into five equal lengths; starting from point A, connect every other subdivision point on the median to points B and C (points F and H); erase labels E and G and segment $\overline{HD}$.



Area of $\triangle AFB$ = Area of $\triangle AFC$ = Area of $\triangle BFH$ = Area of $\triangle CFH$ = Area of $\triangle BHC$

Comments

T 56
Subdividing Triangles
Teacher Notes

T 56
Subdividing Triangles

Task: To develop a method of subdividing triangles into congruent triangles where the number of triangles is greater than or equal to 3.

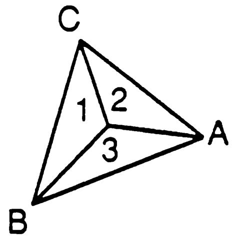
Procedure:
- Construct any $\triangle ABC$.
- Create a method.
- Describe your method.
- State your conjectures.

Drawings & Data

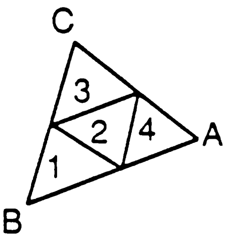
Conjectures

Conjectures

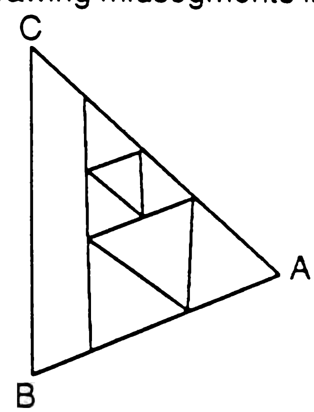
- For n = 3, only in equilateral triangles; draw three medians.



- For n = 4, in any triangle; draw three midsegments.

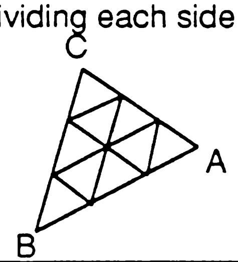


- For $n = k \cdot 4$, k an integer, in any triangle; repeat previous method of drawing midsegments in smaller triangles.



- For $n = k \cdot 12$, k an integer, in any triangle; combine the first two methods.

- For $n = k^2$, k an integer, in any triangle; by parallels and subdividing each side into k parts.



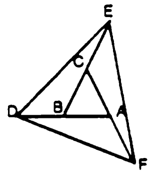
Comments

Extended Sides
Parts A and B
Teacher Notes

T 57
Extended Sides - Part A

Task: To explore relationships in figures formed by extending the sides of triangles by lengths equal to the lengths of the sides.

Procedure:
- Construct an equilateral triangle using the side-side-side option with sides that are 3 units long.
- Extend $\overline{AB}$ from B so that the extension $\overline{BD} \cong \overline{AB}$.
- Extend $\overline{BC}$ from C so that the extension $\overline{CE} \cong \overline{BC}$.
- Extend $\overline{CA}$ from A so that the extension $\overline{AF} \cong \overline{CA}$.
- Draw segments connecting the three endpoints of the extensions.
- State your conjectures.
- Repeat this procedure on two triangles with the following measurements:
  - side lengths 2–2–2
  - side lengths 4–3–3
- State your conjectures.



Drawings & Data

Conjectures

T 57
Extended Sides - Part B

Task: To explore relationships in figures formed by extending the sides of triangles by lengths equal to the lengths of the sides.

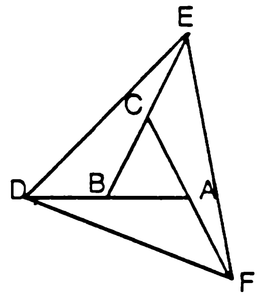
Procedure:
- Construct a triangle using the side-angle-side option so that AB = 3, ∠BAC = 90°, and AC = 2.
- Extend $\overline{AB}$ from B so that the extension $\overline{BD} \cong \overline{AB}$.
- Extend $\overline{BC}$ from C so that the extension $\overline{CE} \cong \overline{BC}$.
- Extend $\overline{CA}$ from A so that the extension $\overline{AF} \cong \overline{CA}$.
- Draw segments connecting the three endpoints of the extensions.
- State your conjectures.

Drawings & Data

Conjectures

Conjectures

Part A



- For any triangle, Area of ΔDBE = Area of ΔECF = Area of ΔFAD

$$\frac{\text{Area of } \Delta DBE}{\text{Area of } \Delta ABC} = \frac{\text{Area of } \Delta ECF}{\text{Area of } \Delta ABC} = \frac{\text{Area of } \Delta FAD}{\text{Area of } \Delta ABC} = 2$$

$$\frac{\text{Area of } \Delta DEF}{\text{Area of } \Delta ABC} = 7$$

- For equilateral triangles:
  $\Delta DBE \cong \Delta ECF \cong \Delta FAD$; corresponding parts are ≅.
  ΔDEF is equilateral.
  Perimeter of ΔDBE = perimeter of ΔECF = perimeter of ΔFAD

Notes

This problem offers good examples of triangles that are <u>not congruent</u> but have <u>equal area</u>.

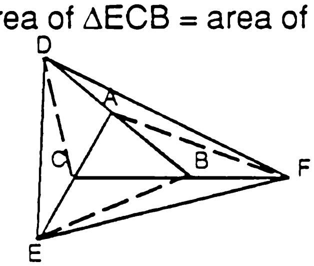
Conjectures

Part B

- For right triangle, 3 units/90°/2 units, the perimeter of ΔECF = perimeter of ΔDAF.

Notes

Because a median divides any triangle into two sections of equal area, the conjectures noted above involving area are true. Simply draw $\overline{DC}$, $\overline{AF}$, and $\overline{BE}$. Then area of ΔCDA = area of ΔABC = area of ΔABF = area of ΔDAF = area of ΔECB = area of ΔEBF = area of ΔECD.



Comments

T 58
Scale Change

Task: To understand the effect of the Scale change option.

Procedure:
- Draw a right ΔABC.
- Measure its sides, angles, and perimeter.
- Record your drawing of the triangle and its measurements in the space provided.
- Use the Scale change option and measure the sides, angles, and perimeter of the new ΔABC.
- Draw this scaled ΔABC below but change the labels on the vertices of the triangle.
- Replace A with A', B with B', and C with C'. $\Delta A'B'C'$ is called the image of ΔABC.
- Repeat the steps above for at least one right, one acute, one obtuse, one isosceles, and one equilateral triangle.
- Record your results.
- State your conjectures.

Drawings & Data

Right Triangle

Acute Triangle

T 58 (page 2)
Scale Change

Obtuse Triangle

Isosceles Triangle

Equilateral Triangle

Conjectures

Conjectures

- ΔABC and $\Delta A'B'C'$ will have corresponding angles with the same measure
- The corresponding sides of ΔABC and $\Delta A'B'C'$ are proportional.

$$\frac{A'B' + A'C' + B'C'}{AB + AC + BC} = \frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}$$

Notes

Once these relationships are established, the definition of similarity between triangles should be introduced together with the Angle-Angle postulate for similar triangles. While working on problems involving similarity, students will need calculators for computing ratios and proportions.

Comments

Scaling with Medians and Altitudes

Teacher Notes

T 59

Scaling with Medians and Altitudes

Task: To understand how a scale change affects lines drawn in a triangle and the area of the triangle.

Procedure:

- Construct an acute ΔABC.
- Draw a median and altitude from one vertex.
- Measure the sides, the median, the altitude, and the area of the triangle.
- Record the drawing of the triangle and its measurements.
- Use the Scale change option.
- Measure the sides, the median, the altitude, and the area of the new triangle.
- Record your results on the new triangle.
- Use letters with "primes" attached to denote labels for vertices (A'B'C').
- Repeat the steps above for at least one obtuse, one isosceles, and one equilateral triangle.
- State your conjectures.

Drawings & Data

Acute Triangle

Obtuse Triangle

T 59 (page 2)

Scaling with Medians and Altitudes

Isosceles Triangle

Equilateral Triangle

Conjectures

Conjectures

- The ratios of the lengths of corresponding medians and the ratios of the lengths of corresponding altitudes are the same as the ratios of the lengths of the corresponding sides of ΔABC and $\Delta A'B'C'$.

Notes

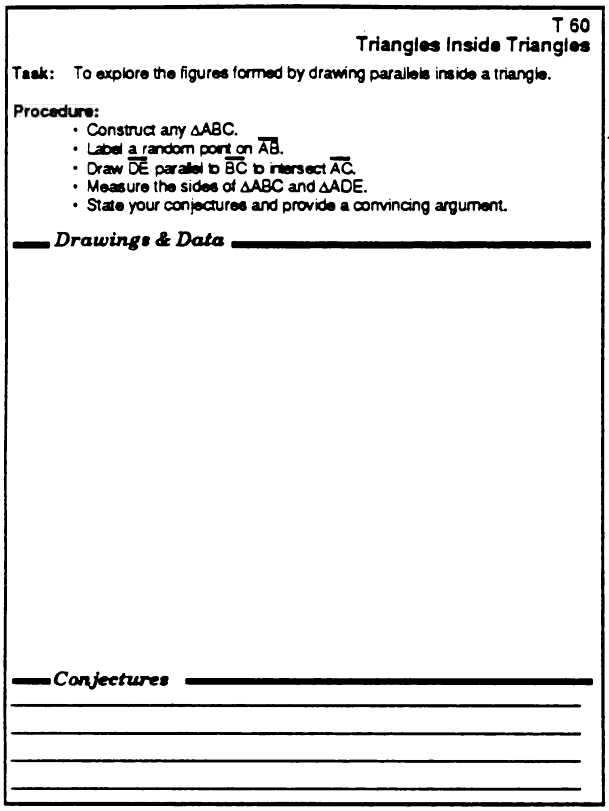
The comparison of corresponding altitudes can be used later to justify why. . .

$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta A'B'C'} = \left[\frac{AB}{A'B'}\right]^2$$

Comments

Triangles Inside Triangles

Teacher Notes



T 60
Triangles Inside Triangles

Task: To explore the figures formed by drawing parallels inside a triangle.

Procedure:
- Construct any ΔABC.
- Label a random point on $\overline{AB}$.
- Draw $\overline{DE}$ parallel to $\overline{BC}$ to intersect $\overline{AC}$.
- Measure the sides of ΔABC and ΔADE.
- State your conjectures and provide a convincing argument.

Drawings & Data

Conjectures

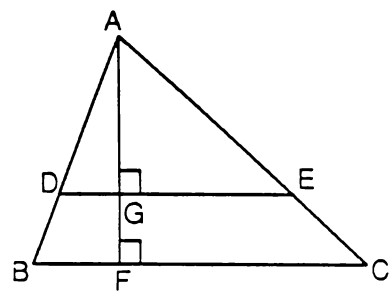
Conjectures

The purpose of this problem is to introduce students to the notion that two triangles can be similar even if one is drawn inside the other. In particular $\Delta ADE \sim \Delta ABC$ because $\overline{DE} \parallel \overline{BC}$ and angles ADE, ABC, and angles AED, ACB form congruent corresponding pairs of angles.

- $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ADE} = \left[\dfrac{AB}{AD}\right]^2 = \left[\dfrac{BC}{DE}\right]^2 = \left[\dfrac{AC}{AE}\right]^2$

Proof

Draw altitudes $\overline{AF}$ and $\overline{AG}$ for ΔADE and ΔABC respectively.



$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ADE} = \frac{\frac{1}{2} \cdot BC \cdot AF}{\frac{1}{2} \cdot DE \cdot AG}$$

$$= \left[\frac{BC}{DE}\right]\left[\frac{AF}{AG}\right].$$

$$= \left[\frac{BC}{DE}\right]\left[\frac{BC}{DE}\right] \text{ since } \frac{AF}{AG} = \frac{BC}{DE}$$

$$= \left[\frac{BC}{DE}\right]^2 = \left[\frac{AB}{AD}\right]^2 = \left[\frac{AC}{AE}\right]^2$$

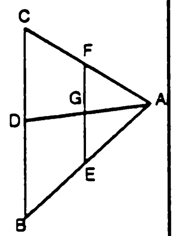
Comments

T 61
Midsegment and Median
Teacher Notes

T 61
Midsegment and Median

Task: To explore the relationship between segments formed by a median and a midsegment.

Procedure:
- Construct any $\triangle ABC$.
- Draw median $\overline{AD}$.
- Draw midsegment $\overline{EF}$ joining sides $\overline{AB}$ and $\overline{AC}$.
- Label the intersection of $\overline{AD}$ and $\overline{EF}$ with point G.
- Measure the segments.
- State your conjectures.
- Provide supporting arguments.



Drawings & Data

Conjectures

What is the relationship between EG and FG?

Provide supporting arguments for your conjectures.

Conjectures

- $EG = FG$

Proof

Since $\overline{EF}$ is a midsegment, $\overline{EF} \parallel \overline{BC}$. Since $\overline{GF} \parallel \overline{DC}$ and $\overline{EG} \parallel \overline{BD}$, we have $\Delta AGF \sim \Delta ADC$ and $\Delta AEG \sim \Delta ABD$ by the AA Postulate.
Now, $\Delta AEF \sim \Delta ABC$ because $\overline{EF} \parallel \overline{BC}$ and the AA Postulate. Thus,

$$\frac{AF}{AC} = \frac{EF}{BC} = \frac{EF}{2 \cdot EF} = \frac{1}{2}; \quad \text{similarly, } \frac{AE}{AB} = \frac{1}{2}$$

Consider $\Delta AGF \sim \Delta ADC$. Then, $\dfrac{GF}{DC} = \dfrac{AF}{AC} = \dfrac{1}{2}$ or $GF = \dfrac{1}{2} \cdot DC$

Similarly, $\dfrac{EG}{BD} = \dfrac{AE}{AB} = \dfrac{1}{2}$ or $EG = \dfrac{1}{2} \cdot BD$

But, since $BD = DC$ ($\overline{AD}$ is a median),
$EG = \dfrac{1}{2} \cdot DC$

Since $GF = \dfrac{1}{2} \cdot DC$ and $EG = \dfrac{1}{2} \cdot DC$,
we have $EG = GF$ and G is the midpoint of $\overline{EF}$.

Comments

T 62
Mystery Triangles - Part A

Task: To explore a set of triangles.

Procedure:
- Construct each triangle listed in the table below and on the following page.
- Complete the table.
- State your conjectures on the following page.

	Triangle Drawings	∠ABC	∠BCA	∠CAB
AB = 2 AC = 3 BC = 4				
AB = 3 AC = 4.5 BC = 6				
AB = 4.5 AC = 6.75 BC = 9				

T 62 (page 2)
Mystery Triangles - Part A

	Triangle Drawings	∠ABC	∠BCA	∠CAB
AB = 10 AC = 9 BC = 8				
AB = 7.5 AC = 6.75 BC = 6				
AB = 5 AC = 4.5 BC = 4				

Conjectures

Conjectures

Part A

- In each ΔABC: $\angle ABC = 46.57°$, $\angle BCA = 28.96°$, and $\angle CAB = 104.48°$.
  Students should discover the above conjecture and <u>recognize</u> that the ratios of the lengths of the corresponding sides of any <u>two</u> triangles are equal. In the fourth set, the ratio is $\frac{3}{2}$; in the second set, the ratio is $\frac{3}{4}$.

Comments

Mystery Triangles
Part B

Teacher Notes

T 62
Mystery Triangles - Part B

Task: To explore a set of triangles.

Procedure:
- Construct each triangle listed in the table below and on the following page.
- Complete the table.
- State your conjectures on the following page.

	Triangle Drawings	AB	BC	AC	∠ACB
∠BAC = 30 AB = 4 ∠CBA = 70					
∠BAC = 30 AB = 5 ∠CBA = 70					
∠BAC = 30 AB = 6 ∠CBA = 70					

Conjectures

Conjectures

Part B

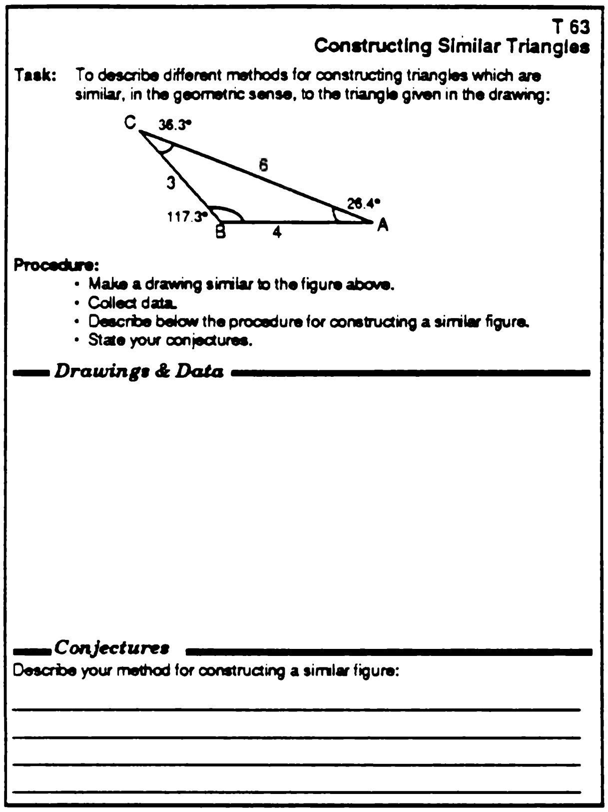
- ∠ACB = 80° in each triangle
- The ratios of the lengths of the corresponding sides of any two triangles are equal.

Notes

The purpose of this exercise is for students to use the Angle-Side-Angle option to generate triangles that are similar to each other. Students should discover that the ratio of the lengths of the corresponding sides of any two triangles will be equal if the two triangles have the same angle measurements.

Comments

T 63
Constructing Similar Triangles
Teacher Notes


T 63
Constructing Similar Triangles

Task: To describe different methods for constructing triangles which are similar, in the geometric sense, to the triangle given in the drawing:

Procedure:
- Make a drawing similar to the figure above.
- Collect data.
- Describe below the procedure for constructing a similar figure.
- State your conjectures.

Drawings & Data

Conjectures

Describe your method for constructing a similar figure:

Conjectures

- Use any length of side with any two angles given in the drawing. (ASA)
- Increase or decrease the lengths of the three sides by the same scale factor. (SSS)
- Same angle and two scaled sides. (SAS)

Notes

The purpose of this project is to help students understand 'similarity' before they have learned the ASA, SSS, SAS similarity theorems. This project might be easy for many students but from some of them, you may get procedures such as: increase each of the sides by one unit or divide each angle by two. Be sure to clarify the meaning of similarity in class discussion before introducing the similarity theorem.

Comments

Dividing Sides Parts A and B

Teacher Notes

T 64
Dividing Sides - Part A

Task: To explore relationships between a triangle and a smaller triangle formed using subdivision.

Procedure:
- Construct any $\triangle ABC$.
- Subdivide $\overline{AC}$ and $\overline{AB}$ into three equal parts.
- Draw $\overline{DF}$.
- Measure the elements of the triangles.
- Repeat the drawing on different triangles.
- State your conjectures about the relationship between $\triangle ABC$ and $\triangle ADF$.

Drawings & Data

Conjectures

T 64
Dividing Sides - Part B

Task: To explore relationships between a triangle and a smaller triangle formed using subdivision.

Procedure:
- Use your construction from Part A:
  - Construct any $\triangle ABC$.
  - Subdivide $\overline{AC}$ and $\overline{AB}$ into three equal parts.
  - Draw $\overline{DF}$.
- Draw a median from vertex A in $\triangle ABC$.
- Draw a median from vertex A in $\triangle ADF$.
- Measure the elements of the triangles.
- Repeat the drawing on three other triangles.
- State your conjectures about the relationship between $\triangle ABC$ and $\triangle ADF$.

Drawings & Data

Conjectures

Conjectures

Part A

- $\Delta AFD \sim \Delta ABC$
- $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta AFD} = 9$
- $\dfrac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta AFD} = 3$

Part B

- A median divides a side in the same ratio as it divides any parallel to that side.

Comments

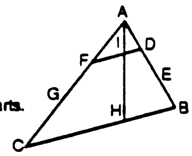
Dividing Sides Part C

Teacher Notes

T 64

Dividing Sides - Part C

Task: To explore relationships between a triangle and a smaller triangle formed using subdivision.



Procedure:

- Use your construction from Part B:
  - Construct any $\triangle ABC$.
  - Subdivide $\overline{AC}$ and $\overline{AB}$ into three equal parts.
  - Draw $\overline{DF}$.
- Label a random point H on $\overline{BC}$.
- Draw $\overline{AH}$ and label point I as the intersection of $\overline{FD}$ and $\overline{AH}$.
- Measure $\overline{BH}$, $\overline{CH}$, $\overline{DI}$, and $\overline{FI}$.
- Find relationships.
- Label another random point on $\overline{BC}$ and repeat the procedure.
- State your conjectures.

Drawings & Data

Conjectures

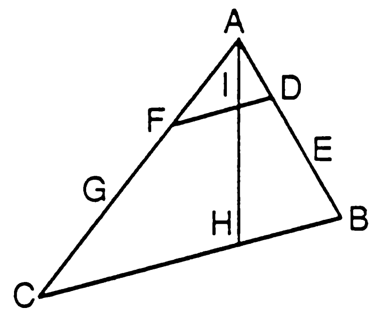
Conjectures

Part C

A segment from a vertex to the opposite side divides the side in the same ratio as it divides any parallel to that side.

Data

CH = 5.84
HB = 4.63
CH/HB = 1.26
FI = 1.95
ID = 1.54
FI/ID = 1.26



Comments

Creating Similar Triangles

Teacher Notes

T 65

Creating Similar Triangles

A line segment inside a triangle can divide a triangle into two similar triangles. In some cases these two triangles are also similar to the original triangle; in other cases, the line segment only creates one triangle which is similar to the original triangle.

Task: To explore conditions under which a line segment divides a triangle into similar triangles.

Procedure:

- Inside any ΔABC draw any line segment using options on the Draw and Label menus.
- Determine if and how many similar triangles are created.
- Record information in the chart below.
- Repeat the construction for different types of triangles and record your findings.
- Repeat the procedure for different types of line segments.
- State your conjectures on the following page.

Description of Segment Drawn	Type of Triangle	One similar triangle	Two similar triangles	Three similar triangles

T 65 (page 2)

Creating Similar Triangles

Conjectures

Conjectures

- A line segment emanates from a vertex and makes an angle of the same size as one of the angles in ΔABC (can be done without the SUPPOSER but as a generalization of the altitude in right triangles): two similar triangles result.
- An angle bisector from the vertex angle in an isosceles triangle or from any angle in an equilateral triangle: two congruent (similar) triangles result.
- One midsegment creates two similar triangles.
- Subdivisions of sides and parallel lines create two similar triangles.
- In a right triangle, an altitude drawn from the right angle creates three similar triangles.

Comments

Similar Triangles That Don't Touch
Parts A and B

Teacher Notes

T 66

Similar Triangles That Don't Touch - Part A

Task: To describe methods for drawing a triangle inside ΔABC that is similar to ΔABC such that the two triangles share no points.

Procedure:

- Draw any ΔABC.
- Draw the similar triangle so that it is located anywhere inside ΔABC.
- Provide data to verify that your methods work.
- State your conjectures.

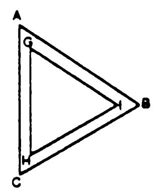
Drawings & Data

Conjectures

T 66

Similar Triangles That Don't Touch - Part B

Task: To describe methods for drawing a triangle inside ΔABC that is similar to ΔABC such that the sides are equidistant from the corresponding sides of ΔABC.



Procedure:

- Draw any ΔABC.
- Draw a similar triangle inside ΔABC such that the sides are equidistant from the corresponding sides of ΔABC.
- Provide data to verify that your methods work.
- State your conjectures.

Drawings & Data

Conjectures

Conjectures

Part A

- The most general method to draw a similar triangle inside ΔABC is
  (1) take any point inside ΔABC,
  (2) subdivide the segments that connect the vertices of ΔABC to the random point into the same number of subdivisions, and
  (3) draw a triangle by connecting three "appropriate" points from the subdivisions.

Part B

- (1) Draw the three angle bisectors of ΔABC,
  (2) subdivide the segments that connect the vertices of ΔABC to the point of intersection of the angle bisectors into the same numbers of subdivisions,
  (3) draw a triangle by connecting the three "appropriate points" from the subdivisions.

Notes

A related problem deals with constructing a similar triangle that is rotated 180° from the original triangle. Another connected problem is how to draw a "frame" outside the original triangle.

Comments

T 67
Trigonometric Ratios

The data in the chart below represents the measurements of one side (AB = 5) and three angles in a set of right triangles where ∠A = 90°. The trigonometric functions sine, tangent, and cosine are defined for ∠B.

Task: To explore trigonometric functions in right triangles.

Procedure:
- Construct each right triangle listed below.
- Compute the ratios CA/CB, CA/AB, and AB/CB.
- Record the ratios.
- State your conjectures.

∠A	∠B	∠C	AB	sin ∠B (CA/CB)	tan ∠B (CA/AB)	cos ∠B (AB/CB)
90°	10°	80°	5	.17	.18	.98
90°	15°	75°	5	.26	.27	.97
90°	20°	70°	5			
90°	25°	65°	5			
90°	30°	60°	5			
90°	40°	50°	5			
90°	45°	45°	5			
90°	50°	40°	5			
90°	60°	30°	5			
90°	70°	20°	7			
90°	80°	10°	7			
90°	85°	5°	4			

Conjectures

State conjectures about the behavior of the ratios (the values of the trigonometric functions) as ∠B and ∠C either increase or decrease in size.

Conjectures

- For non-90° angles in right triangles, the values of sine and cosine are all between zero and one.
- The values of tangent are all positive.
- sin ∠B = cos (90°– ∠B) and cos ∠B = sin (90°– ∠B)
- For angles less than 30°, the sine behaves almost linearly.

∠A	∠B	∠C	AB	sin ∠B (CA/CB)	tan ∠B (CA/AB)	cos ∠B (AB/CB)
90°	10°	80°	5	.17	.18	.98
90°	15°	75°	5	.26	.27	.97
90°	20°	70°	5	.34	.36	.94
90°	25°	65°	5	.42	.47	.91
90°	30°	60°	5	.5	.58	.87
90°	40°	50°	5	.64	.84	.77
90°	45°	45°	5	.71	1	.71
90°	50°	40°	5	.77	1.19	.64
90°	60°	30°	5			
90°	70°	20°	7			
90°	80°	10°	7			
90°	85°	5°	4			

Comments

T 68
Constant Perimeter
Part A
Teacher Notes

T 68
Constant Perimeter - Part A

The chart below lists the lengths of the sides of four triangles. Each triangle has a perimeter of 21 units.

Task: To explore triangles that have the same perimeter.

Procedure:
- Draw each triangle using the side-side-side option and the measurements in the chart below.
- Use the Repeat option "on previous shape" to scan the four triangles.
- Record any changes that you observe as you scan the triangles.
- Choose one of the triangles.
- Draw its inscribed circle.
- Draw a radius to one of the tangent points.
- Using the Repeat option, scan the four triangles with their inscribed circle and radii.
- Record any changes that you observe.
- State your conjectures.

AB	AC	BC	Triangles
7	7	7	
7	4	10	
8	4	9	
6	7	8	

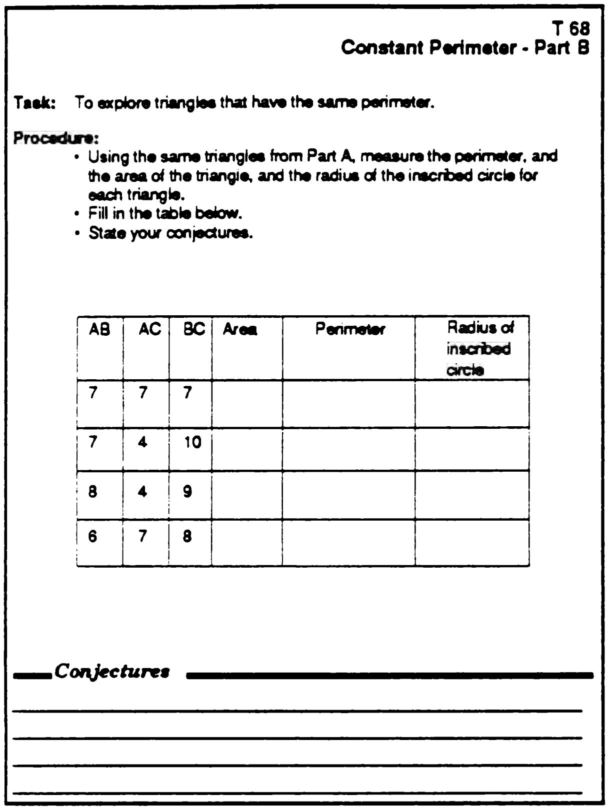
Conjectures

Conjectures

- Triangles with the same perimeter can have different areas.
- The 6–7–8 and 8–4–9 triangles appear to have greater areas than the 7–4–10 triangle.
- The equilateral triangle has the maximum area.
- The larger the area of the triangle, the larger the radius.
- The larger the area of the triangle, the larger the circle.

Comments

T 68
Constant Perimeter
Part B
Teacher Notes



T 68
Constant Perimeter - Part B

Task: To explore triangles that have the same perimeter.

Procedure:
- Using the same triangles from Part A, measure the perimeter, and the area of the triangle, and the radius of the inscribed circle for each triangle.
- Fill in the table below.
- State your conjectures.

AB	AC	BC	Area	Perimeter	Radius of inscribed circle
7	7	7			
7	4	10			
8	4	9			
6	7	8			

Conjectures

Conjectures

AB	AC	BC	Area	Perimeter	Radius of inscribed circle
7	7	7	21.23	21	2.02
7	4	10	10.92	21	1.04
8	4	9	16.00	21	1.52
6	7	8	20.34	21	1.94

- The ratio between the area of ΔABC and the radius of the inscribed circle is constant.
- $$\frac{\text{Area of } \Delta ABC}{\text{Perimeter of } \Delta ABC} = \frac{\text{Radius of the inscribed circle}}{2}$$

  or Area of ΔABC = (Perimeter of ΔABC) $\cdot\, r \cdot \frac{1}{2}$

Proof

Area of ΔABC = Area of ΔADC + Area of ΔBDC + Area of ΔABD

$$\frac{1}{2}(AC \cdot DG) + \frac{1}{2}(BC \cdot DF) + \frac{1}{2}(AB \cdot DE)$$

$$(AC + BC + AB) \cdot \frac{r}{2} \text{ since } DG = DF = DE = r,$$

$$= (\text{Perimeter of } \Delta ABC) \cdot \frac{r}{2}$$

Comments

Constant Area

Teacher Notes

T 69
Constant Area

The table below contains four triangles whose areas are 10 square units.

Task: To explore triangles that have the same area.

Procedure:
- Draw each triangle using the side-angle-side option and the measurements in the chart below.
- Find the perimeter of each triangle and the length of the radius of the inscribed circle.
- State your conjectures.

Side	Angle	Side	Drawing	Area	Perimeter	Radius
5	90°	4.00				
6	90°	3.33				
7	90°	2.86				
10	90°	2.00				

Conjectures

Conjectures

Side	Angle	Side	**Drawing**	Area	Perimeter	Radius
5	90°	4.00		10	15.4	1.30
6	90°	3.33		10	16.2	1.23
7	90°	2.86		10	17.4	1.15
10	90°	2.00		10	22.2	0.90

- As one leg increases in length, the other leg decreases in length.
- As one leg increases in length, the perimeter increases and the radius of the inscribed circle decreases.
- The product of the perimeter and radius is constant.
- The larger the perimeter, the smaller the radius.

Notes

Embedded in this problem are several principles of relations at different levels: the relation between perimeter and area, the notion of maximum and minimum area for a given perimeter; the maximum and minimum perimeter for a given area.

The exact ratio: (2 x Area)/Perimeter = Radius is a traditional problem in trigonometry. However, if the properties of the center of an inscribed circle are known, then the problem is trivial to prove by computation of the sum of the areas of the original triangle and three sub-triangles.

Comments

Center of the Circumscribed Circle

Teacher Notes

T 70

Center of the Circumscribed Circle

Task: To investigate relationships in a figure formed by triangles that are circumscribed by circles.

Procedure:

- Construct any ΔABC.
- Circumscribe ΔABC with a circle having center D.
- Repeat the construction on different types of triangles.
- Record the diagrams.
- Make measurements.
- State your conjectures.

Drawings & Data

Conjectures

The location of the center D relative to the triangle:

____

____

The relative size of the circle and the triangle:

____

____

The relationship of point D to ΔABC:

____

____

Conjectures

- Suppose $\angle BAC = 90°$. As $\angle BAC$ increases in size, the center D of the circumscribed circle moves into the exterior of the triangle.
- In a right triangle the point D is on the hypotenuse.
- As $\angle BAC$ increases in size, the area of ΔBAC occupies less of the circle.
- An equilateral triangle occupies the maximum area of the circle.
- The point D is equidistant from the vertices of ΔABC.

Notes

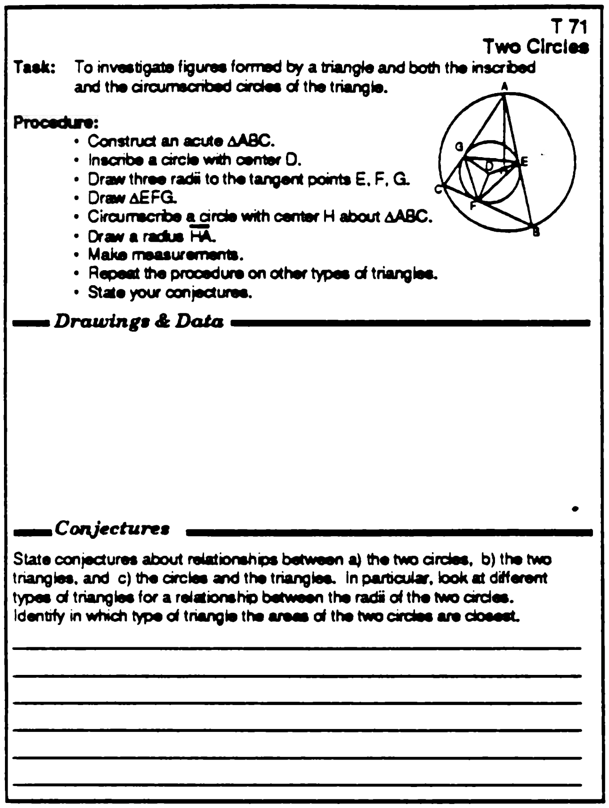
This project emphasizes the option to collect data and impressions from visual changes before taking measurements. In many cases, the relationship between two (or more) shapes is clear enough so that conjectures can be based on visual data and then be checked quantitatively and deductively.

In the case of circumscribed circles, the different locations of the center and the relative size (area) of the circle and the triangle is clear and can help students think inductively.

Comments

Two Circles

Teacher Notes



T 71
Two Circles

Task: To investigate figures formed by a triangle and both the inscribed and the circumscribed circles of the triangle.

Procedure:
- Construct an acute ΔABC.
- Inscribe a circle with center D.
- Draw three radii to the tangent points E, F, G.
- Draw ΔEFG.
- Circumscribe a circle with center H about ΔABC.
- Draw a radius $\overline{HA}$.
- Make measurements.
- Repeat the procedure on other types of triangles.
- State your conjectures.

Drawings & Data

Conjectures

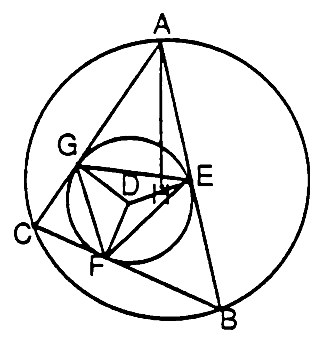
State conjectures about relationships between a) the two circles, b) the two triangles, and c) the circles and the triangles. In particular, look at different types of triangles for a relationship between the radii of the two circles. Identify in which type of triangle the areas of the two circles are closest.

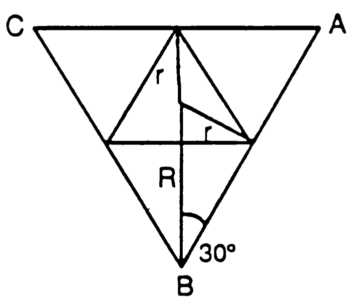
Conjectures

- The ratio between the radii of the two circles is always greater than or equal to 2. This ratio is minimal for equilateral triangles.
- $$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta EFG} = 2 \cdot \frac{\text{radius of circumscribed circle}}{\text{radius of inscribed circle}}$$

$$= 2 \cdot \frac{R}{r}$$

Data
HA = 5.57
DE = 2.6
HA/DE = 2.14
A:ABC = 36.89
A:EFG = 8.62
ABC/EFG = 4.28





In an **equilateral triangle**,

- H and D are the same point, HD = 0
- $$\frac{\text{radius of circumscribed circle}}{\text{radius of inscribed circle}} = 2$$
- because $\sin 30° = \frac{1}{2}$, and

  in Δ ADE $\sin 30° = \frac{DE}{AH} = \frac{r}{R}$, so $\frac{R}{r} = 2$
- $$\frac{\text{Area of circumscribed circle}}{\text{Area of inscribed circle}} = 4$$

  because $\frac{\pi R^2}{\pi r^2} = \frac{R^2}{r^2} = \left(\frac{R}{r}\right)^2 = 2^2 = 4$

Notes

This project could serve as a review and summary for the properties of inscribed and circumscribed circles.

Comments

T 72
Midpoints and Circumscribed Circles
Parts A and B
Teacher Notes

T 72
Midpoints and Circumscribed Circles - Part A

Task: To investigate the relationship between the circumscribed circle of a triangle and the circle which circumscribes the smaller triangle formed by connecting the midpoints of the sides of the original triangle.

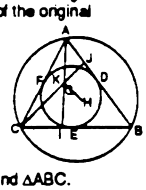
Procedure:
- Construct any acute $\triangle ABC$.
- Bisect the three sides producing points D, E, and F.
- Draw a circumscribed circle around $\triangle DEF$ and $\triangle ABC$.
- Make measurements.
- Repeat the procedure for other triangles and collect data.
- State your conjectures about the relationships between the two circles.

Drawings & Data

Conjectures

T 72
Midpoints and Circumscribed Circles - Part B

Task: To investigate the relationship between the circumscribed circle of a triangle and the circle which circumscribes the smaller triangle formed by connecting the midpoints of the sides of the original triangle.



Procedure:
- Use your construction from Part A:
  - Construct an acute $\triangle ABC$.
  - Bisect the three sides producing points D, E, and F.
  - Draw a circumscribed circle with center G around $\triangle DEF$ and one with center H around $\triangle ABC$.
- Draw any two altitudes in $\triangle ABC$ and label their intersection with point K.
- Make measurements.
- Repeat the procedure for other triangles and collect data.
- State your conjectures about the relationship among points G, H, and K.

Drawings & Data

Conjectures

What is the relationship between points:

G - The center of the circumcircle of $\triangle DEF$. ________

H - The center of the circumcircle of $\triangle ABC$. ________

K - The intersection point of the altitudes. ________

Part A

- $\dfrac{\text{Radius of circumscribed circle for } \triangle ABC}{\text{Radius of circumscribed circle for } \triangle DEF} = 2$
- If $\triangle ABC$ is equilateral, the circumscribed circles are concentric.
- If $\triangle ABC$ is a right triangle, the circumscribed circles are tangent at the vertex of the right angle.
- If $\triangle ABC$ is obtuse, the circumscribed circles intersect in two points.

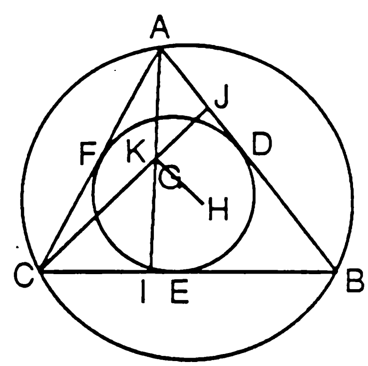
Part B

Conjectures

- G is the midpoint of $\overline{HK}$.
- H, G, and K are collinear.
- If $\triangle ABC$ is equilateral, H, G, and K are all the same point.

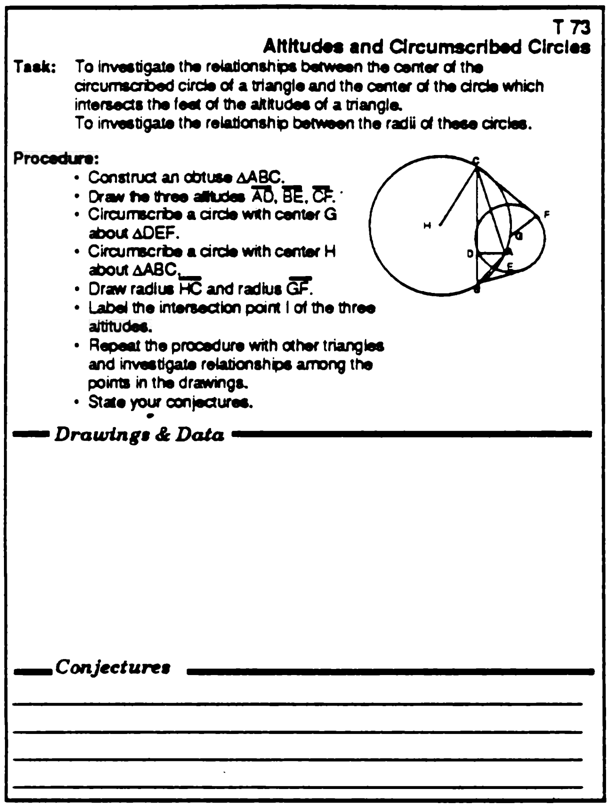
Data

HG = 1.38
GK = 1.38
<HGK = 180



Comments

T 73
Altitudes and Circumscribed Circles
Teacher Notes


T 73
Altitudes and Circumscribed Circles

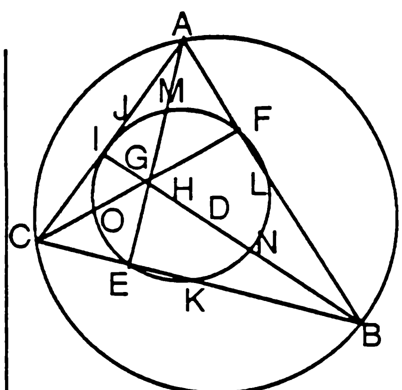
Task: To investigate the relationships between the center of the circumscribed circle of a triangle and the center of the circle which intersects the feet of the altitudes of a triangle.
To investigate the relationship between the radii of these circles.

Procedure:
- Construct an obtuse ΔABC.
- Draw the three altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$.
- Circumscribe a circle with center G about ΔDEF.
- Circumscribe a circle with center H about ΔABC.
- Draw radius $\overline{HC}$ and radius $\overline{GF}$.
- Label the intersection point I of the three altitudes.
- Repeat the procedure with other triangles and investigate relationships among the points in the drawings.
- State your conjectures.

Drawings & Data

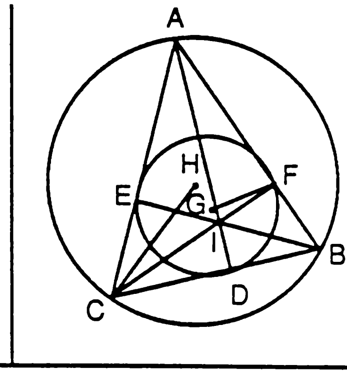
Conjectures

Data
BN = 3.99
NG = 3.99
BN/NG = 1
AM = 1.45
MG = 1.45
AM/MG = 1
CO = 2.67
OG = 2.67
CO/OG = 1
BK = 5.35
KC = 5.35
BK/KC = 1
CJ = 3.84
AJ = 3.84
CJ/AJ = 1
AL = 4.85
BL = 4.85
AL/BL = 1



Conjectures

Data
HC = 5.39
GF = 2.7
HC/GF = 2
IG = 1.44
GH = 1.44
IG/GH= 1
<IGH = 180



- $\dfrac{HC}{GF} = 2$, or

$$\frac{\text{Radius of circumscribed circle about } \Delta ABC}{\text{Radius of circumscribed circle about } \Delta DEF} = 2$$

- IG = HG where I is the intersection of the altitudes, $\overline{GF}$ is the radius of the circumcircle for ΔDEF, and $\overline{HC}$ is the radius of the circumcircle for ΔABC
- I, H, and G are collinear
- For equilateral triangles, I, H, and G coincide
- For isosceles triangles, I, H, and G lie on a symmetry line

Notes

Problems 71, 72, and 73 can be used as a resource for drawing a nine point circle.

The following is the method for drawing the nine point circle. Circumscribe circle D about acute ΔABC. Draw altitudes from vertices A and C and label the intersection point G. Label the midpoint of DG with point H. H is called the center of the nine point circle which passes through the midpoints of the sides of ΔABC, the foot of each altitude, and the midpoints between each vertex of ΔABC and point G (the intersection of the altitudes).

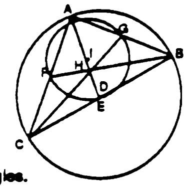
Comments

T 74
Medians and Circumscribed Circles
Teacher Notes

T 74
Medians and Circumscribed Circles

Task: To investigate the relationship between the centers of the circumscribed circle of a triangle and the circle which intersects the feet of the medians of a triangle.

Procedure:



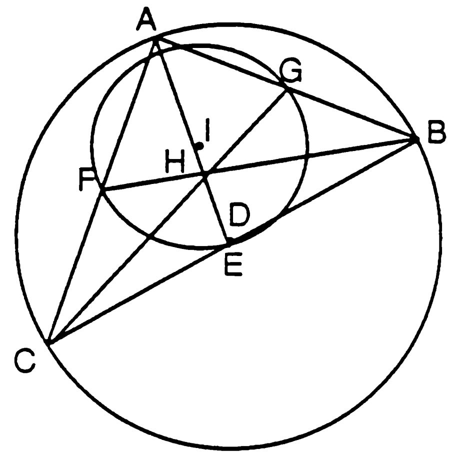
- Construct ΔABC.
- Draw the circumscribed circle D about ΔABC.
- Draw the three medians: $\overline{AE}$, $\overline{BF}$, and $\overline{CG}$.
- Label their intersection point H.
- Draw the circumscribed circle I about ΔEFG.
- Investigate relationships between the three points D, H, and I.
- Repeat this procedure on other types of triangles.
- State your conjectures.

Drawings & Data

Conjectures

Conjectures

- $\dfrac{HD}{HI} = 2$



Comments